

THE
PHYSICAL SOCIETY
OF
LONDON.

PROCEEDINGS.

VOLUME XXI.—PART VII.

FEBRUARY 1910.

LONDON:

TAYLOR AND FRANCIS, RED LION COURT, FLEET STREET.
1910.

PHYSICAL SOCIETY OF LONDON.

1909-10.

OFFICERS AND COUNCIL.

President.

C. CHREE, Sc.D., LL.D., F.R.S.

Vice-Presidents

WHO HAVE FILLED THE OFFICE OF PRESIDENT.

PROF. G. G. FOSTER, D.Sc., LL.D., F.R.S.
PROF. W. G. ADAMS, M.A., F.R.S.
PROF. R. B. OLIFTON, M.A., F.R.S.
PROF. A. W. REINOLD, M.A., F.R.S.
SIR ARTHUR W. RÜCKER, M.A., D.Sc., F.R.S.
SIR W. DE W. ABNEY, R.E., K.C.B., D.C.L., F.R.S.
SHELFORD BIDWELL, M.A., LL.B., Sc.D., F.R.S.
PRIN. SIR OLIVER J. LODGE, D.Sc., LL.D., F.R.S.
PROF. SILVANUS P. THOMPSON, D.Sc., F.R.S.
R. T. GLAZEBROOK, D.Sc., F.R.S.
PROF. J. H. POYNTING, Sc.D., F.R.S.
PROF. J. PERRY, D.Sc., F.R.S.

Vice-Presidents.

W. DUDELL, F.R.S.
PROF. A. SCHUSTER, Ph.D., F.R.S.
S. SKINNER, M.A.
W. WATSON, D.Sc., F.R.S.

Secretaries.

W. R. COOPER, M.A.
82 Victoria Street, S.W.

S. W. J. SMITH, M.A., D.Sc.

Imperial College of Science and Technology, South Kensington.

Foreign Secretary.

PROF. S. P. THOMPSON, D.Sc., F.R.S.

Treasurer.

PROF. H. L. CALLENDAR, M.A., LL.D., F.R.S.
49 Grange Road, Ealing, W.

Librarian.

W. WATSON, D.Sc., F.R.S.
Royal College of Science.

Other Members of Council.

A. CAMPBELL, B.A.
W. H. ECOLES, D.Sc.
A. GRIFFITHS, D.Sc.
J. A. HARKER, D.Sc.
PROF. C. H. LEES, D.Sc., F.R.S.
T. MATHER, F.R.S.
A. RUSSELL, M.A., D.Sc.
PROF. E. RUTHERFORD, D.Sc., F.R.S.
F. E. SMITH.
R. S. WHIPPLE.

I. *On a Want of Symmetry shown by Secondary X-Rays.*
 By W. H. BRAGG, M.A., F.R.S., Elder Professor of
Mathematics and Physics in the University of Adelaide,
 and J. L. GLASSON*.

[From "Transactions of the Royal Society of South Australia,"
 vol. xxxii., 1908.]

ON the assumption that the Röntgen rays consist of æther pulses it has been shown by J. J. Thomson ("Conduct. of Electr. through Gases," p. 323) that it is possible to account for the existence of secondary Röntgen rays by assuming that the primary pulses set in motion electrons over which they pass, and cause them to become new centres of radiation. If the electron easily follows the guiding force of the primary pulse, then the secondary radiation resembles the primary in quality. But if the electron is hampered by attachments to other portions of the atom to which it belongs, then the new pulse has not the same quality as the old; the time of motion of the electron is dragged out, and the pulse produced is softer.

Now, if an electron becomes in this way a centre of radiation the intensity of the secondary effect must be symmetrical about the line of motion of the electron. In particular, the intensity of the secondary radiation must be symmetrical about a plane passing through the electron perpendicular to the primary ray, since this ray contains the line of motion referred to. This deduction forms an integral part of Thomson's theory of secondary Röntgen radiation, and its truth has been assumed in calculations intended to show that experimental results are in agreement with theory. Barkla proves the same deduction in a paper published in the Philosophical Magazine of February 1908.

Now it has recently been shown (Bragg and Madsen, Trans. Roy. Soc. S.A., May 1908) that the cathode radiations excited by γ rays show a very marked want of symmetry about the plane normal to the exciting ray; and again (Madsen, Trans. Roy. Soc. S.A., July 1908) that

* Read April 23, 1909.

there is a similar want of symmetry in respect to the secondary γ rays. The γ rays and X-rays resemble one another so closely in all their known properties, that it is fairly safe to assume any effect found to be true of the one kind to be true also of the other kind, though perhaps to a different degree. In this case, indeed, Cooksey ('Nature,' April 2, 1908) has already shown that the secondary cathode radiations excited by X-rays are not at all symmetrical about the normal plane, the emergence rays being greater than the incidence, as in the case of the γ rays.

It remained, therefore, to examine the secondary X-rays excited by primary X-rays; and the experiments described in this paper were made with that object. We find that in general want of symmetry does exist, that it is sometimes very pronounced, and that is in keeping with expectation based on Madsen's study of the secondary γ rays. Hard γ rays show a very large difference between the quantities of emergence and incidence radiation; for soft γ rays the difference is smaller. Since X-rays are to be looked on as a very soft form of γ rays, the difference should be smaller still; and this is what we have found to be the case.

The general form of the apparatus which we have used is shown in fig. 1. Variations of the upper portion of it are shown in figs. 2 and 3. A small pencil of X-rays passed upwards through apertures in lead plates at A and B, and then along the axis of the ionization-chamber and out into the open. In our first experiments the upper part of the apparatus was arranged as in fig. 3. The primary rays did not pass through the effective part of the ionization-chamber, being separated therefrom by the cylindrical screen SS, which could be made of various thicknesses and various materials. But if a thin sheet of any substance was laid over the hole at B, secondary X-rays spread out therefrom, and some passed through the screen SS, and caused a deflexion in the electrometer. The difference between the deflexions (*a*) without and (*b*) with the sheet at B was taken as a measure of the emergence secondary X-ray radiation. When the sheet was removed from B, and the same or a similar sheet placed in the plane of the top of the screen so as to be struck from below by the primary rays, then the

measure of the incidence secondary radiation was obtained as the difference between the deflexions (*a*) without and (*c*) with the sheet so placed.

In this way it was easy to show that the expected want of symmetry actually existed, particularly with aluminium,

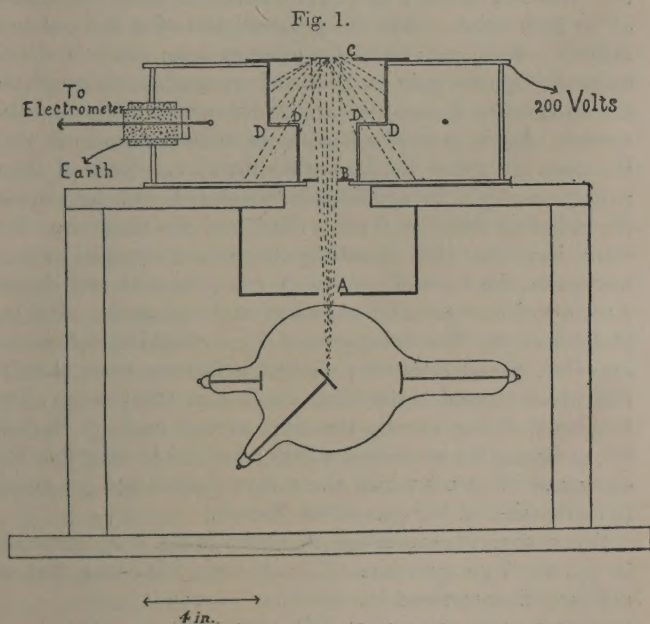


Fig. 2.

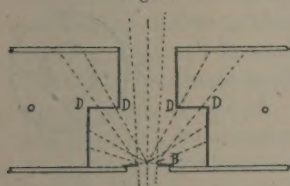
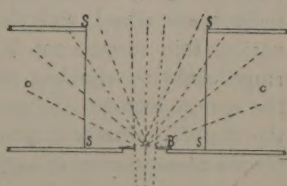


Fig. 3.

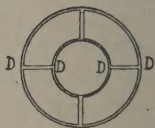


celluloid, or paper as the radiators, substances of small atomic weight. But the experiments were open to some extent to the objection that *a* was too large compared with *b-c*, and that possibly the excess of emergence over incidence

was an apparent effect due to actual variations of a under different circumstances. The current a was, in fact, due to several causes. There was a small natural ionization leak even when the X-rays were not acting; there was an effect due to primary X-rays which had penetrated the walls of the chamber, though they were made of zinc one-eighth of an inch thick. But the greatest part of a was due to a diffusion of soft rays about the primary beam, much of which came through the hole at B at such an angle as to penetrate the screen SS; it could be largely cut out by thickening the screen. Again, part of a was due to radiation returned from the open air above the ionization-chamber. Some of these radiations might be appreciably interfered with by placing the radiating sheet at B or at the top of the chamber. We were, however, able to satisfy ourselves by special experiments that the want of symmetry was quite real, and that as a matter of fact no valid objection could be made. But we abandoned the first arrangement for a second which, as we expected, would show the want of symmetry more clearly, and which proved better than the first in every way. The first method was exactly the same as that used by Madsen in examining the secondary γ rays; but it was clear that the enormous difference which these rays showed was not going to be repeated in the case of the X-rays.

Our new arrangement was, as shown in fig. 1, or, inverted, in fig. 2. Two cylinders of brass, each 2 in. long, but of different diameters—4 in. and 2 in.—were fixed to a connecting piece DD, shown in plan in fig. 4. The latter resembled a light brass wheel with four spokes, and various thin screens cut in the form of flat rings could be attached to it, filling up all the spaces between the spokes. In fig. 1 the double cylinder is shown as arranged for the measurement of incidence secondary radiations; the radiating sheet was placed at C, supported by a sheet of celluloid lying flat on the top of the cylinder. A hole was cut in the centre of the celluloid sheet big enough to allow the primary beam to pass through without touching the edges; and a fluorescent screen was used to make sure

Fig. 4.



that this was the case. The radiating sheets were of thin metal, about $1\frac{1}{2}$ in. square. In fig. 2 the cylinder is shown as arranged for the measurement of emergence secondary radiations: it hardly requires further explanation.

We expected that this arrangement would show up the want of symmetry better than the former, because the portions of the emergence and incidence beams under comparison would be more nearly normal to the plate. Looking upon the radiations as material, we should naturally expect the intensity of the secondary radiation to decrease gradually as its direction increased in inclination to the forward direction of the primary ray. The emergence rays lie, in inclination, between 0° and 90° ; the incidence between 90° and 180° . In our first arrangement we compared the emergence rays between about 40° and 90° , with the incidence rays between about 90° and 140° . There should be a larger ratio of emergence to incidence with the newer arrangement, since the emergence rays between about 30° and 50° would be compared with the incidence between about 130° and 150° . This proved to be the case; the improvement was considerable. Again, with the new arrangement, the current with no radiator in position became relatively far smaller. For example, when the radiator was Al, .4 mm. thick, and the absorbing screen DD of tinfoil (two thin sheets), the currents with and without the radiator at B in fig. 1 caused deflexions of 86 and 26 mm. in ten seconds respectively; the currents with and without the radiator at B in fig. 2 were 220 and 35 respectively. There could be very little error, therefore, in taking the incidence and emergence radiations as 60 and 185 respectively; and the want of symmetry is beyond doubt.

It should be observed that the emergence radiation can never be shown to an unfair advantage in these experiments, and is often at a disadvantage, for the radiator, when placed as in fig. 2, cuts down the very primary rays to which the secondary radiation is due. It is not difficult to show that if the thickness of the radiator is so adjusted as to give the maximum emergence current (it can of course be too thick or too thin), then the ratio of this maximum to the maximum incidence current (which can be obtained simply by making

the radiator thick enough) is only $2/e$ of the true ratio of emergence to incidence; provided that the secondary rays are as penetrating as the primary, and that we are considering homogeneous radiations. But if, other conditions being the same, the secondary rays are less penetrating than the primary, then the ratio, as found, is more nearly correct, and is very nearly so when the secondary rays are much less penetrating than the primary, as, for example, when we are considering secondary cathode rays due to X- or γ rays.

We have made a large number of measurements by the method described above, using the following metal sheets as radiators :—Pt, weight per square cm., $\cdot 0150$ gr. : Sn, $\cdot 0096$ gr. ; Cu, $\cdot 0083$ gr. ; Fe, $\cdot 0077$ gr. ; Al, $\cdot 105$ gr. ; celluloid, $\cdot 20$ gr. As screens we have used various thicknesses of Sn, Cu, and Al.

The proportion of emergence to incidence radiation differs considerably for the different radiators, but is much the same for different screens or different thicknesses of screen, except that the proportion tends to increase slightly as the screen is made thicker; and the tendency is most pronounced in the case of those metals which give out a quantity of soft secondary radiation. For example, Fe and Cu show little difference between incidence and emergence radiations until the screen is so thick that only a small fraction of either of the radiations can pass through. The results vary somewhat with the state of the bulb; and since these variations are comparable with those which are met with on changing the nature of the screens, we are not now in a position to discuss smaller variations in detail. We must content ourselves with quoting a few results in order to show the want of symmetry, which is a persistent effect. When, for example, two tinfoils were used as screen (weight per square cm. of each, $\cdot 0056$), we obtained the following figures, which represent movements of the scale in mm. during 10 secs. :—

Radiator	Sn.	Cu.	Fe.	Al.
Emergence Current	176	140	39	185
Incidence Current	122	119	15	60

With four tinfoils the figures were :—

Radiator	Sn.	Cu.	Fe.	Al.
Emergence Current	143	24	23	116
Incidence Current	87	1	0	34

Again, using a copper screen .002 cm. thick, we found :—

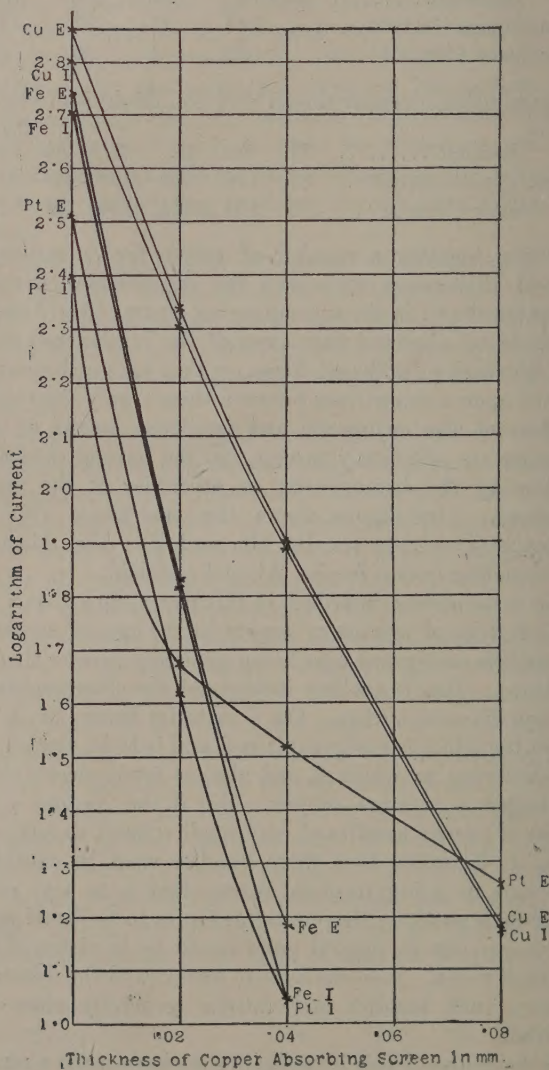
Radiator	Pt.	Sn.	Cu.	Fe.	Al.	Cellu- loid.
Emergence Current ...	86	140	361	118	80	138
Incidence Current ...	65	104	364	118	32	93

Putting together a number of results for Cu screens of different thicknesses we obtain the logarithmic curves of absorption shown in the accompanying figures (figs. 5 and 6). It should be observed that some of the results thus shown were obtained at different times, so that too much must not be built upon a comparison between them ; only the relative positions of the emergence and incidence curves of each substance are sufficiently correct, and the form of each curve as showing the homogeneity or otherwise of the various radiations. One figure shows the emergence (E) and incidence (I) curves for Pt, Cu, and Fe ; the other the corresponding curves for Sn, Al, and celluloid.

The experiments described in this paper show that a very marked want of symmetry occurs in the case of secondary X-rays, the emergence rays being generally greater than the incidence. This is another instance of the close parallelism between X- and γ rays. On a material theory of X- and γ rays the effect is easily explained, and is to be classed with the scattering to which β , and also, as lately shown clearly by Geiger, α rays are subject. But if the X- and γ rays consist of energy bundles of very small volume, as suggested by J. J. Thomson, then these bundles must be capable of deflexions in going through atoms—that is to say, swung out of their paths by the electrical forces to be found within the atoms, just as neutral pairs would be in virtue of their electrical fields. It seems hard to understand the distinction between such bundles and entities generally classed as material.

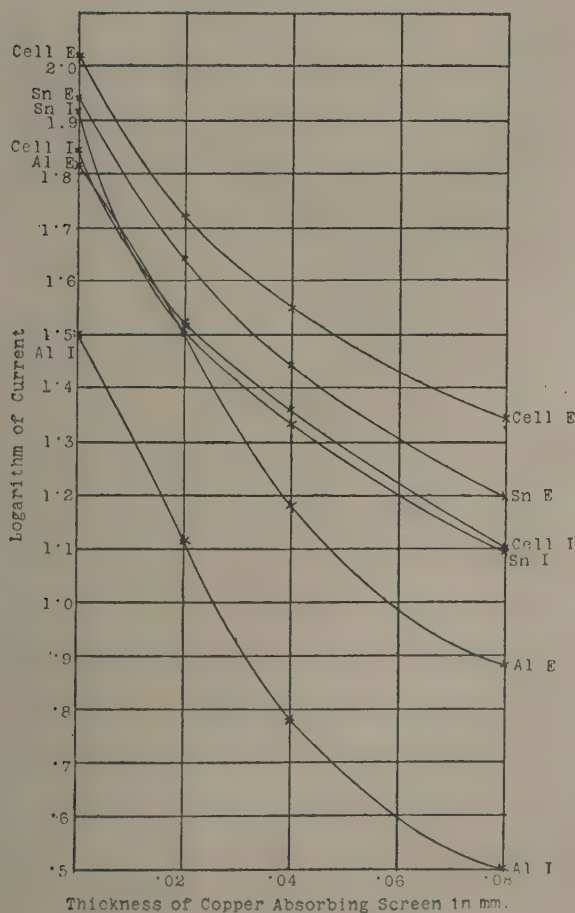
In the course of this investigation we have made a number

Fig. 5.



of experiments on the quantities and qualities of the secondary radiations. This subject has been fully treated by Barkla, some of whose recent papers have not yet reached us, and

Fig. 6.



any discussion we gave might be merely a duplication of part of his inquiry. There is, however, one point to which we should like to refer.

Very hard γ rays follow a density law of absorption, treating all atoms alike, except in respect to weight. Soft γ rays are not independent of atomic groupings of matter, and are far more strongly absorbed by heavy atoms than by light, after allowance has been made for weight. The same is generally true of X-rays; but in the case of very soft X-rays there is a tendency to revert to the density law again. For instance, X-rays that have passed through the glass of the bulb are soft to copper, silver, tin, and so on, but hard to aluminium, carbon, and low atomic weight generally. No doubt those rays which are soft to such light atoms have already been absorbed by the glass. But secondary X-rays from most substances are softer than anything emerging from the bulb and contained in the primary ray. The difference is not very great when the absorption is measured with the aid of screens made of substances of the higher atomic weights, because to these the primary rays are soft already. But if the screens are made of aluminium, still more of filter-paper, the difference now seems to be very great, for the secondary rays are soft even to low atomic weights. For example, in one experiment, a sheet of copper weighing .018 gr. per square cm. caused a drop of .401 in the logarithm (to base 10) of the primary rays, and only of .447 in the case of the emergence secondary rays from copper, of .645 in the case of platinum rays, and .805 of iron rays. But when four filter-papers weighing .02 gr. per square cm. were used as screen, the drop in the case of the primary rays was .010—only one-fortieth of the drop caused by a copper screen of nearly equal weight. In the case of the secondary rays, however, the same screen caused a drop in the case of copper rays of .100, platinum rays .053, and iron rays of .188—that is to say, for these soft rays the filter-papers are much more nearly on an equality with copper, weight for weight, than they were for hard rays. It is interesting to bear this in mind when considering the very large quantities of secondary ionization which some substances seem to give. The ionization is always measured in air, which of course consists of atoms not very different in weight from those contained in filter-papers. Consequently primary rays, and secondary rays which differ

very little from the primary, are very penetrating to air, and cause relatively small ionizations therein. But secondary rays from Cu and Fe are softened so much as to bring them within reach, so to speak, of air, which rapidly converts them into cathode rays, so that there is a very large ionization. For the cathode rays produced from these secondary rays have probably but little less energy than those produced from the primary; the speed of the cathode ray does not differ very greatly with the penetration of the primary X-ray, so far as experiments have shown. The very large secondary radiations, which some substances appear to give, therefore, owe their magnitude largely to the fact that the air in which they are measured is sometimes ten to twenty times as favourable to them as to the primary rays which produced them. In this way we may account to some extent for the startling results obtained by Crowther in the case of arsenic and bromine (Phil. Mag. Nov. 1907).

DISCUSSION.

Prof. C. H. LEES said that Prof. Bragg had given a lucid account of his theories of γ and X rays. His researches would make physicists more careful in accepting the æther-pulse theory. He asked if it was likely that better means would be devised to discriminate between various forms of γ and X rays than dividing them into "hard" and "soft" radiations. He thought many discrepancies could be attributed to this want of discrimination.

Mr. C. A. SADLER pointed out that whatever lack of symmetry might exist in the emergence and incidence secondary X radiations from a plate of a substance which was a source of scattered primary radiation, Professor Bragg's own results conclusively proved that such lack of symmetry did *not* exist when the plate was a source of homogeneous radiation. If then it was a necessary condition of Professor Bragg's theory that such lack of symmetry should exist with secondary X radiations, we must either conclude that the theory here breaks down or that these homogeneous radiations are not X radiations as usually understood. It was to be noted also that the measured lack of symmetry (ignoring the lack of symmetry in the case of homogeneous beams, which can be shown to be only apparent) in the most pronounced cases was small compared with those obtained with γ rays.

Prof. BRAGG, referring to the remarks of Prof. Lees, said that for precision the actual speed of all electrons ought to be measured. Instead of measuring the speed the penetrating power might be determined.

LI. *Transformations of X-Rays.* By CHARLES A. SADLER,
M.Sc., Oliver Lodge Fellow, University of Liverpool.*

WHEN a primary beam of Röntgen radiation falls upon any substance, secondary Röntgen rays are emitted, the character of which depends both upon the nature of the substance and upon the particular kind of primary beam used. With reference to the latter it has been found that variations in the intensity of the primary beam produce no perceptible change in the character of the resulting secondary, the sole controlling factor appearing to be the degree of "hardness" of the primary †.

It has been shown ‡ that if the radiating substance be an element of low atomic weight—as hydrogen, oxygen, or carbon—it emits a radiation similar in penetrating power to the primary producing it; its penetrating power varying with that of the producing primary. This type of radiation, which will be referred to as a "scattered" radiation, may be considered as produced by an acceleration of one or more electrons in the atom of the radiating substance, due to the action of forces in the primary pulse.

From an element of greater atomic weight than that of calcium (40)—and possibly from other elements of lower atomic weight under very penetrating primary beams—the emitted radiation has been shown to consist of scattered radiation, and superposed upon this, a type of radiation which is characteristic of the radiating element. The penetrating power of this radiation is a constant quantity peculiar to the substance and is independent of the penetrating power of the primary producing it. Moreover, this radiation appears to be entirely homogeneous in character, and experiments point to the conclusion that the radiating electrons producing this type of radiation are no longer appreciably under the influence of the forces in the primary pulse.

This homogeneous radiation is only produced when the

* Read April 23, 1909.

† Barkla, *Phil. Mag.* June 1906, pp. 812-828

‡ Barkla & Sadler, *Phil. Mag.* Oct. 1908 pp. 550-584.

penetrating power of the primary is greater than that of the homogeneous radiation characteristic of the radiator.

From the group of elements Cr-Ag the ionization produced by the homogeneous portion of the secondary radiation, emitted when any of its members is subjected to a sufficiently penetrating primary, is many times greater than that produced by the scattered portion—in the case of copper as radiator this ratio is as high as 150 : 1.

The homogeneous radiation from chromium is very “soft” much softer indeed than any ordinary primary beam, and from chromium down to silver and probably beyond, the penetrating power of the characteristic radiation increases with increase of atomic weight; the radiation from silver being many times more penetrating than that from chromium.

One of the chief difficulties experienced in the investigation of X-ray phenomena has been the heterogeneity of the primary beams hitherto available. Even where devices are adopted to ensure that the current through the X-ray bulb used as a source of primary rays is uni-directional and of nearly constant strength, the primary so obtained consists of a mixture of constituents of different penetrating power; so there remains the difficulty of ascertaining which particular constituents of the composite beam are principally concerned in producing the phenomena under investigation.

It was thought that useful information concerning the nature of X-rays might be obtained if the homogeneous rays previously mentioned were used to excite tertiary radiation in different substances.

Sagnac has shown that the tertiary X-rays from metals excited by secondary X-rays are more easily absorbed than the exciting rays.

Previous experiments* had shown that if two substances A and B be taken, each of which is found to emit a homogeneous radiation when a suitable primary beam falls upon it, the homogeneous radiation from A being more penetrating than that from B, then if a homogeneous beam from A be passed through a thin plate of B, tertiary radiation is

* Barkla & Sadler, *Phil. Mag.* Oct. 1908, pp. 550-584.

excited in B by the radiation from A, while if the process is reversed it is found that the radiation from B excites *no* tertiary radiation in A.

These phenomena were examined in greater detail in the experiments described below.

It was found that no trace of a homogeneous tertiary radiation could be detected from aluminium when subjected to any of the homogeneous radiations from the group of metals Cr-Ag, and the amount of scattered radiation produced was extremely small when compared with the secondary incident beam.

Advantage was taken of these facts, and the primary and secondary beams were passed through tubes of thick aluminium of rectangular cross-section. This enabled the apparatus to be arranged compactly with comparatively short distances between the anticathode of the X-ray bulb and the secondary radiator, and between the secondary and tertiary radiators respectively, a condition essential to secure that the ionization produced by the tertiary rays in a suitable ionization chamber should be sufficiently intense to ensure accurate readings in a reasonably short time, and that the direct tertiary radiation should produce an ionization large compared with that produced by stray secondary and tertiary rays from the surrounding air and neighbouring screens.

The general arrangement of the apparatus is indicated in Plan by fig. 1.

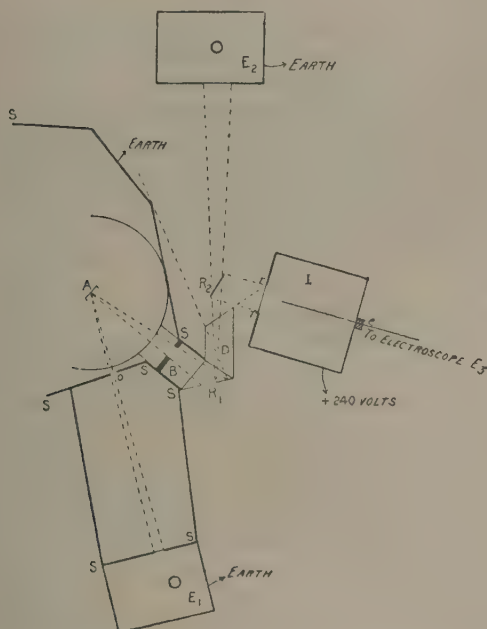
A rectangular brass tube B lined with aluminium $\cdot 2$ cm. thick was fitted into an aperture in the lead screen SS surrounding the X-ray bulb emitting the primary rays, and the bulb was so placed relatively to this tube that the axis of the tube passed through the centre of the anticathode A normally.

The rays from A passing through this tube impinged on the radiator R_1 consisting of a rectangular plate of the metal, the secondary rays from which were to be studied. A portion of the secondary rays so produced passed down the rectangular brass tube D lined as before with $\cdot 2$ cm. aluminium, and a plate of any substance placed in this secondary beam provided a source of tertiary rays.

The intensity of the primary beam was measured by

allowing a narrow pencil of rays proceeding from A through a small aperture O in the lead screen SS to enter an electro-scope E_1 of the ordinary Wilson type through an opening

Fig. 1.



covered with tissue-paper and aluminium-foil in the wall of the electro-scope.

The deflexions of the gold-leaf were observed by means of a microscope fitted with a scale in the eyepiece.

The intensity of the secondary beam was measured by means of a similar electro-scope E_2 placed in the path of the secondary beam passing down the tube D (R_2 remove!) as indicated in the Plan; the size of the aperture in the wall of this electro-scope being 3×2 cms.

It was found that by carefully shielding the electroscopes E_1 and E_2 from draughts and sudden changes of temperature very reliable readings could be obtained, the motion of the gold-leaf being absolutely dead-beat.

Preliminary experiments also showed that with a given radiator R_1 in position (R_2 being removed) the ratio of the deflexions in E_1 and E_2 when the bulb had reached a steady state rarely varied by more than 1 per cent. during a series of readings.

The ordinary Wilson type of electroscope was found to be quite unsuitable as a means of measuring the tertiary radiation; the type of electroscope described by R. T. Beatty in his paper on "Secondary Röntgen Radiation in Air" * was finally adopted. Briefly described, this consisted of a brass case having two sliding quadrants insulated from the case and charged to potentials of +240 and -240 volts respectively; an insulated gold-leaf hung vertically between them and was connected to the wire e , which projected into and was insulated from the ionization-chamber I, which itself was insulated and charged to +240 volts. An adjustable compensating-chamber insulated and charged to -240 volts eliminated the normal ionization in the chamber I. The sensitiveness of the electroscope was adjustable by means of the sliding quadrants.

With the order of sensitiveness required in all the experiments described, the motion of the gold-leaf was dead-beat, and a series of readings of the ratio of deflexions in electroscopes E_1 and E_2 showed that with deflexions up to 20 scale-divisions in E_2 this could be obtained with an accuracy of 2 per cent. with certainty.

A calibration of electroscopes E_1 and E_2 showed that the value of a deflexion of a scale-division was the same within 1 per cent. throughout the range of scale employed.

The sensitiveness of E_3 was adjusted and then maintained constant throughout a given series of readings, and in all readings a constant deflexion of E_3 was used, thus eliminating the effect of the slight change in value of a scale-division which was found to exist in different parts of the scale.

Preliminary experiments showed that the secondary radiation from air together with other stray effects were very small compared with the readings obtained when metallic radiators were employed.

* R. T. Beatty, *Phil. Mag.* Nov. 1907, pp. 604-614.

The portion of the radiator R_1 exposed to the primary beam was limited to that opposite to the tube D by means of a suitable stop placed in the tube B as indicated in the diagram.

Nature of the Tertiary Rays.

It was perhaps reasonable to expect that the tertiary radiation emitted by any member of the group Cr-Ag when subjected to a more penetrating homogeneous beam from the same group would be identical in character with that emitted as secondary radiation by the same substance when excited by a suitable primary.

Direct experiments were carried out to test how far this was true, and for this purpose pure iron was chosen as the tertiary radiator.

The radiator consisted of a small rectangle 3 cms. high by 2 cms. broad, formed of pure iron wire interlaced so as to expose as large a surface as possible to the exciting beam. This was placed in the position indicated by R_2 , the centre of the radiator was at the intersection of the axis of the tube D with the normal from the centre of the aperture to the ionization-chamber I, the plane of the radiator making equal angles with these directions.

As a secondary radiator in the position R_1 a plate of pure copper was used.

The aperture YY of the ionization-chamber I in these experiments was 3 cms. high by 2 cms. broad, and the distance from the centre of the aperture to the centre of R_2 , 4 cms.

Owing to the obliquity of some of the tertiary rays, it was evident that the absorption coefficients, obtained by studying the absorption by thin plates of different substances placed parallel to the aperture YY in the path of the beam, would be greater than would have been the case had it been possible to utilize a *pencil* of tertiary rays.

A control experiment conducted with a fairly soft primary beam falling upon the same iron radiator similarly situated before an aperture of the same size as that in the screen YY in an electroscope of the Wilson type placed in the secondary beam from the iron, gave an increase in the value of the

absorption coefficient by an aluminium plate .00297 cm. thick of 6 per cent. over the value found when a narrow pencil of secondary radiation was used.

The absorption coefficients of the tertiary beam from iron were then determined by thin sheets of aluminium, iron, copper, and zinc. The values so obtained are compared with those obtained when the same absorbers were used with the secondary rays in the control experiment in the following table.

TABLE I.

Absorber.	Value of $\frac{\lambda}{\rho}$ for Secondary Rays.	Value of $\frac{\lambda}{\rho}$ for Tertiary Rays.
Al (.00297 cm.) ...	93.8	94.2
Fe (.00315 cm.) ...	69.1	69.1
Cu (.00298 cm.) ...	101.0	102.5
Zn (.00132 cm.) ...	119.2	120.0

It will thus be seen that within the limits of experimental errors the penetrating power of the tertiary beam is identical with that of a similar secondary beam from the same substance.

The Homogeneity of the characteristic Tertiary Radiation.

The tertiary beam from iron excited by the secondary homogeneous beam from copper was now cut down by thin aluminium sheets to test its homogeneity.

It was to be expected, even were the beam perfectly homogeneous, that after cutting down by a few plates the beam would appear slightly more penetrating, for the more oblique rays would suffer extinction to a greater extent than those passing through the absorber in a perpendicular direction.

It was not found possible to test for homogeneity to an exhaustive limit owing to the smallness of the readings in the later stages, but the results obtained show that for all

practical purposes the beam may be regarded as homogeneous.

The results are tabulated below:—

TABLE II.

Iron as Tertiary Radiator ; Copper as Secondary Radiator.

Amount previously absorbed by Aluminium.	Subsequent absorption by a sheet of Al (.00297) cm. thick.
None.	55.6 per cent.
55.6 per cent.	55.7 „
89.3 „	55.5 „
94.0 „	55.0 „

Previous experiments have shown that associated with the homogeneous secondary radiation from a metal of the group Cr—Ag is a small proportion of scattered radiation, the relative ionizations produced by the homogeneous and scattered portions being about 150 : 1.

If the secondary beam from iron be absorbed by say .0104 cm. aluminium, then while the homogeneous portion will be absorbed to the extent of about 90 per cent. the scattered portion will only be absorbed by about 30 per cent., giving an absorption of the whole beam of 89.6 per cent. (the absorption being measured by the relative diminution in ionization) ; so that the relative ionizations produced by the residual homogeneous and scattered portions respectively will now be as 21.4 : 1, and a sheet of .0104 aluminium would now only absorb about 87.4 per cent. of the whole beam, and this increase in penetrating power would become more and more pronounced as further absorptions took place.

In a corresponding case of the tertiary beam from iron excited by copper radiation, the amount of scattered copper radiation in the beam if present at all will not be present to so great an extent since the copper radiation will not penetrate to anything like the same depth in the iron as an ordinary primary radiation, though on the other hand the ionization produced by beams of scattered primary radiation

and of scattered copper radiation conveying equal amounts of energy per second through unit volume of air, will not be equal; the scattered copper radiation being more easily absorbed will produce the greater ionization.

No direct evidence has yet been obtained that when the normal tertiary radiation from iron is excited by homogeneous radiation from copper, any of the copper radiation itself is scattered by the iron; its presence in the beam would be difficult to detect for a small percentage of copper radiation in the beam would produce a very minute change in its absorption coefficients. The figures given in Table II. therefore do not preclude the possibility of a small percentage of the ionization being due to scattered copper radiation.

*Independence of the Penetrating Power of the
Exciting Radiation.*

Experiments were made to test whether the penetrating power of the tertiary radiation emitted by a substance depended in any way upon the penetrating power of the exciting secondary beams. In the first test, iron was chosen as the tertiary radiator, copper and arsenic as the secondary radiators. The radiation from arsenic being about twice as penetrating as that from copper.

In the second test chromium was taken as the tertiary radiator, and iron, copper, and arsenic as secondary radiators; the radiations from each of these substances being more penetrating than that from chromium, the radiation from arsenic being about four times as penetrating as that from iron.

The results are tabulated below :—

TABLE III.
Iron as Tertiary Radiator.

Secondary Radiator.	Absorption coefficient of the Secondary Radiation by Aluminium.	Percentage Absorption of the Tertiary Radiation by (.00297 cm.) Al.
Copper	128.9	55.6
Arsenic	60.7	55.4

TABLE IV.
Chromium as Tertiary Radiator.

Secondary Radiator.	Absorption coefficient of the Secondary Radiation by Aluminium.	Percentage absorption of the Tertiary Radiation by (.00297 cm.) Al.
Iron	239	75.1
Copper	128.9	75.3
Arsenic	60.7	75.1

From these results it will be seen that the penetrating power of the tertiary radiations is uninfluenced, so far as can be measured, by variations in the penetrating power of the exciting secondary beams; they further show that if any secondary radiation is scattered by the tertiary radiator, the ionization it produces is small compared with that produced by the tertiary radiation.

This result is analogous with that obtained with secondary beams*.

It has thus been shown that the characteristic tertiary radiation from iron is identical with the characteristic secondary radiation from iron in its penetrating power, its homogeneity, and in its independence of variations in the penetrating power of the exciting radiation.

Similar results were obtained with copper as a tertiary radiator.

Connexion between the Secondary and Tertiary Radiators.

A series of experiments was now undertaken in which the several members of the group of metals Cr-Ag were used as secondary radiators, the tertiary radiators being also chosen from among the earlier members of the same group.

By referring to column I, Table V. it will be seen that the absorption coefficients by aluminium of the radiations from the metals of this group indicate a wide range of penetrating powers, so that it was reasonable to expect that for each tertiary radiator there would be at least some secondary

* Barkla & Sadler, *Phil. Mag.* Oct. 1908, pp. 550-584.

radiations sufficiently penetrating to cause it to emit its characteristic radiation.

The method of experimenting was as follows:—The secondary radiators successively placed in the position R_1 (see fig. 1) were subjected to the primary rays proceeding from the anticathode A, R_2 being temporarily removed. The ratio of the deflexions of the gold-leaves in the electroscopes E_2 and E_1 obtained from the microscope readings was determined in the case of each radiator; let r_1 be the value of this ratio and r_1' be the value of the ratio with no metallic radiator in the position R_1 , the air in its neighbourhood acting as a source of secondary radiation.

Next, the particular tertiary radiator under examination was placed in the position R_2 (as previously defined), and the secondary radiators were again in turn subjected to the primary rays. Ionization was now produced in the chamber I and deflexions of the gold-leaf in electroscope E_3 were obtained. Let the value of the ratio of the deflexions of the gold-leaves in E_3 and E_1 be r_2 , and when the tertiary radiator was removed and the air alone in its neighbourhood acted as a source of tertiary rays, let the corresponding value of the ratio be r_2' .

The tertiary radiator R_2 was then replaced and the ratio r_2 again found, and then finally removing R_2 the readings for the ratio r_1 were repeated.

Working with an X-ray bulb, having an auxiliary spark-gap, it was found that in the steady state the two sets of values for r_1 and likewise those for r_2 showed agreement to within 2 per cent.

In no case was the air-effect r_1' more than 1 per cent. of the value of r_1 , in most cases less than $\frac{1}{4}$ of 1 per cent.; and in no case was the air-effect r_2' more than 2 per cent. of the value of r_2 when the characteristic homogeneous radiation was excited.

In those cases where the penetrating power of the secondary beam did not exceed that of the characteristic radiation from the tertiary radiator employed, the air-effect r_2' became of considerable importance, being as high as $\frac{1}{4}$ of r_2 in some cases. The author hopes to obtain more reliable data in these particular cases in the course of further experiments.

It will be seen later, however, that an accurate knowledge of these particular data is not essential in the present investigation.

It was estimated that if in all other cases $\frac{1}{2}$ of the air-effect was taken in addition to the normal leakage, and this subtracted in each case from the direct effect when the metallic radiator was in position, the resulting ratios finally obtained would be accurate to at least 2 per cent. If a denote the ratio of the normal leakage in any given time in the electroscope E_2 to the deflexion in E_1 in the same time, with a primary beam as during the actual experiment, then since the normal leakage is compensated for in E_3 , the ratio of the ionizations in the electroscopes E_3 and E_2 due to the homogeneous radiations

$$= \frac{r_2 - \frac{1}{2} r_2'}{r_1 - [\frac{1}{2} r_1' + a]} = R \text{ (say).}$$

If the absorption coefficient of the homogeneous radiation from a given tertiary radiator by Al be denoted by λ_2 , and that of the exciting secondary radiation by λ_1 , then as long as $\lambda_1 =$ or is $> \lambda_2$ the value of R is quite small and approximately constant. As λ_1 decreases through the value λ_2 we get a rapid increase in the value of R , and for a comparatively small subsequent increase in penetrating power of the secondary beam, values of R 30 to 40 times as big as the previous steady values are obtained; this increase corresponding to the excitation of the characteristic tertiary radiation.

Let us consider the tertiary radiation emitted normally from a given radiator of area S upon which a uniform parallel beam of secondary rays is incident normally.

Let us define a quantity k , such that the fraction of the energy of the secondary beam passing normally through a thin layer δx of the tertiary radiator which is transformed into tertiary radiation is $k\delta x$. Thus, if I be a measure of the energy passing normally per second through unit area of the tertiary radiator at a depth x below the surface, the energy transformed per second in a layer $\delta x = Ik\delta x$.

Now Barkla* has shown that when this homogeneous

* Barkla, Phil. Mag. Feb. 1908, pp. 288-296.

type of radiation is excited, it is practically evenly distributed in all directions : consequently, the energy passing per second as tertiary radiation from the layer δx through the tissue-paper-covered window of an electroscope of the Wilson type (I being constant over the whole area of the radiator)

$$= \frac{\omega}{4\pi} SIk\delta x e^{-\lambda_2 x}, \quad . \quad . \quad . \quad . \quad (1)$$

where λ_2 is the absorption coefficient of the tertiary radiation by the material of which the tertiary radiator is composed, and ω is the mean solid angle subtended by the aperture of the electroscope at all parts of the radiating area.

But if I_0 is a measure of the energy incident normally per second on unit area of the tertiary radiator at the surface,

$$I = I_0 e^{-\lambda_1 x},$$

where λ_1 is the absorption coefficient of the secondary beam by the material of which the tertiary radiator is composed, and the whole energy passing into the electroscope per second from the tertiary radiator

$$= \frac{\omega}{4\pi} SI_0 k \int_0^\infty e^{-(\lambda_1 + \lambda_2)x} . dx \quad . \quad . \quad . \quad . \quad (2)$$

$$= \frac{\omega}{4\pi} SI_0 k \frac{1}{\lambda_1 + \lambda_2} . \quad . \quad . \quad . \quad . \quad (3)$$

If now we remove the tertiary radiator, and place the electroscope previously used in the path of the secondary beam, with the centre of its tissue-paper-covered window occupying the position previously occupied by the centre of the radiator, and with its plane perpendicular to the axis of the secondary beam, the energy passing per second through the window, if its area is A , is $I_0 A$.

Therefore the ratio of the energy in the tertiary beam passing per second into the electroscope in the first position to the energy of the secondary beam passing per second into the electroscope in its second position

$$= \frac{\omega}{4\pi} \frac{S}{A} \cdot \frac{k}{\lambda_1 + \lambda_2} . \quad . \quad . \quad . \quad . \quad (4)$$

There is considerable evidence that the ionization produced in a given volume of air by a beam of Röntgen radiation is

approximately proportional to the absorption of that radiation by the air. It has been found* also that the ratio of the absorption coefficients of homogeneous beams of different penetrating powers by any two substances, *e. g.* carbon and aluminium, in which no radiation of the homogeneous type is excited by the beam under consideration, is a constant. Also it has been found that the absorption of a beam of Röntgen rays by any substance depends only upon the quantity of matter present and not upon its state of aggregation.

It is assumed on the basis of these results, that the ionizations produced in a given volume of air by homogeneous beams of different penetrating powers are proportional to the absorption coefficients of these beams by carbon or aluminium.

Making this assumption, we have the ratio of the ionizations produced in the electroscope by the tertiary and secondary beams

$$= \frac{\omega}{4\pi A} \cdot \frac{k}{\lambda_1 + \lambda_2} \cdot \frac{\beta}{\alpha} = \frac{S_2}{S_1} \text{ (say) } \dots (5)$$

where α and β are the absorption coefficients by aluminium of the secondary and tertiary beams respectively. From (5) we find that k

$$= \frac{S_2}{S_1} \cdot \frac{\alpha}{\beta} \cdot \frac{A}{S} \cdot \frac{4\pi}{\omega} \cdot (\lambda_1 + \lambda_2).$$

In the above calculations we have considered the special case of normal incidence and emergence, but, since we are dealing with radiators sufficiently thick to absorb the whole of the incident radiation, it is easy to show that the results (4), (5), (6) are quite general for oblique incidence and emergence, where the incident and emergent beams make equal angles with the normal to the radiating surface.

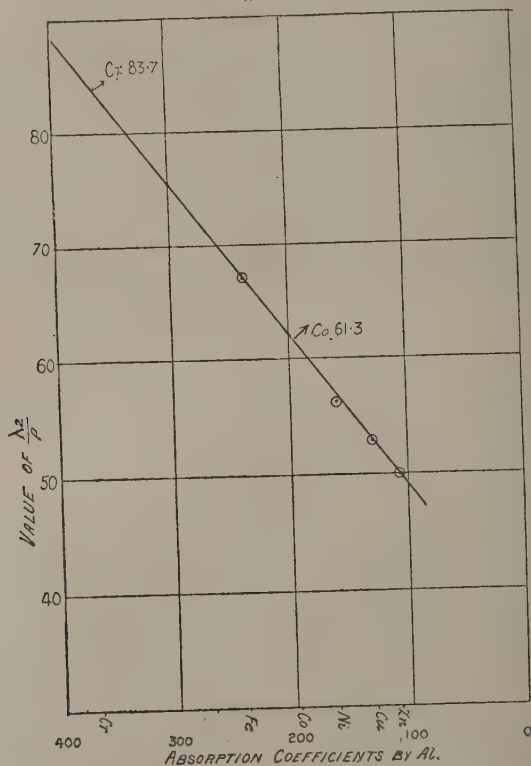
In order to deduce values of k from the experimental data, it is necessary to know the values of α , β , λ , and λ_2 for each of the tertiary radiators employed. α and β are readily obtained for the whole series and likewise λ_1 and λ_2 for Zn, Cu, Ni, and Fe. In the case of Cr and Co, being unable to obtain thin plates of these substances, the values for λ_1 and

* Barkla & Sadler, *Phil. Mag.* May 1909.

λ_2 could not be obtained directly but were deduced by the following method.

If the values of $\frac{\lambda_2}{\rho}$ (where λ_2 is the absorption coefficient by a substance of its own characteristic radiation and ρ its density) for the substances Fe, Ni, Cu, and Zn are plotted

Fig. 2.



as ordinates, against the absorption coefficients for the radiations from these substances by Al as abscissæ, it is found that the points obtained lie on a straight line, as shown in fig. 2.

The values of $\frac{\lambda_2}{\rho}$ for Co and Cr deduced from this graph are Co (61.3), Cr (83.7).

Moreover, a regular relationship is found to exist between the ratios of λ_1 for the radiations from consecutive members of the group Cr-Se for each of the absorbers Zn, Cu, Ni, and Fe. Taking the values of the absorption coefficients for any three of these absorbers for the radiations from the group Cr-Se, values for the coefficients for the fourth absorber could be deduced, starting with a value of λ_2 as a basis and using these observed relationships. In no case was the discrepancy between the experimental and deduced values greater than 2 per cent.

In a similar manner the remaining values of the absorption coefficients for Co and Cr were deduced, taking the values of λ_2 obtained above as a basis for calculation in each case.

TABLE V.

RADIATORS.	Values of λ used in calculation of k .							Values of λ' used in calculation of the ratio $\frac{k}{\lambda'}$.					
	ABSORBERS.												
	Al.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.
Chromium...	367	544											
Iron	239	2500	514					2150					
Cobalt	193.2	2076	521	545				1785	102				
Nickel.....	159.5	1730	2440	584	482			1490	2090	134.0			
Copper	128.9	1478	2080	2560	537	474		1279	1798	2194	150		
Zinc	106.3	227	1715	2172	2275	497	361	1072	1485	1875	1936	120.1	
Arsenic	60.7	755	1040	1333	1422	1575	1464	666	915	1161	1232	1356	1260
Selenium ...	51.0	652	903	1159	1211	1340	1258	577	800	1014	1055	1157	1080
Silver.....	6.75					214	195.1					189.2	170.0

It will be seen later that it is desirable to know what fraction of the absorption of the exiting radiation by the material of the radiator is directly concerned in the transformations of energy which are taking place during the

passage of the secondary beam through the substance of the tertiary radiator.

It has been found* that if the values of λ_1 for the radiations from the group Cr to Ag for any absorber, such as silver, in which no homogeneous radiation is excited by the radiations from any member of the group, be plotted as ordinates against the corresponding values of λ_1 for aluminium, in which also no homogeneous radiation is excited, as abscissæ; the points so obtained lie on straight lines, and these straight lines pass through a common point.

Similarly, if we plot as ordinates the values of λ_1 for any of the early members of the group Cr-Ag as absorber, in which the homogeneous type of radiation is excited by the radiation from those members of the group of higher atomic weight; it has been found, that up to the point corresponding to the atomic weight of the absorber, the relation is also linear, and if the straight line be produced, it passes through the common point previously mentioned. The values of λ_1 for the radiation from the radiators of higher atomic weight are abnormally increased, the increase being attendant upon the excitation of tertiary radiation.

This increase is discussed in the paper on "The Absorption of X Rays" previously referred to.

In the actual experiments to determine the values of R with the various combinations of secondary and tertiary radiators, it was found necessary to alter the position of the quadrants of the tertiary electroscope from time to time, owing to a slight shift of zero, due to small progressive variations in the applied potentials. In this way small changes in the sensitiveness of the instrument were introduced. The sensitiveness could, however, be maintained constant during a period of three or four hours during any given day.

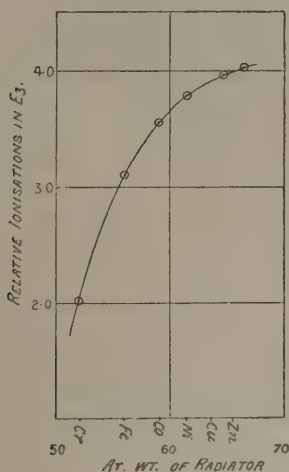
In order to obtain true relative values for the ionizations, the secondary radiation from arsenic was made to fall upon each of the tertiary radiators placed successively in the position R_2 , the sensitiveness of the electroscope remaining constant during this series of observations. Values of R were obtained in this way for Zn, Cu, Ni, and Fe as tertiary radiators, the

* Barkla & Sadler, Phil. Mag. May 1909.

surfaces in each case being brightly polished, of the same size, and thick enough to totally absorb the incident beam. These values of R plotted as ordinates against the atomic weight of the radiators as abscissæ, were found to lie on a smooth curve (see fig. 3).

In the case of Cr and Co, being unable to obtain them pure in the form of flat plates, radiators were made in the following manner. A flat plate of aluminium, of the same size as the required radiator, was covered with a thin layer of

Fig. 3.



adhesive (previously tested and found to emit no detectable radiation of the homogeneous type), and then the metals, finely powdered, were pressed into this layer, the surface being covered with powdered metal. A sample radiator prepared with pure iron filings in this way emitted a tertiary radiation indistinguishable (by absorption tests) from that produced from a plate of the metal. This, however, is quite to be expected since practically the whole of the tertiary radiation from iron is obtained from a layer .003 cm. thick.

But the value of R for a prepared plate of iron filings is a little less than that for a flat polished plate of iron of equal size placed in an identical position, owing to the irregularities

of the surface reducing the effective area. Three equal prepared plates of Co, Fe, and Cr and a polished plate of iron were therefore used, the powders being as nearly as possible equally fine, and the relative values of R obtained. From a comparison of the two values of R for iron, values of R for polished plates of Co and Cr could be deduced, and these were found when plotted to lie almost exactly upon the curve in fig. 3.

From the set of values thus obtained the true relative values of R for the complete series of experiments previously described were deduced and are given below in Table VI.

TABLE VI.

SECONDARY RADIATORS.	Values of R . calculated from the <i>total</i> Ionization in the Ionization-chamber I.						Values of R' . calculated from the <i>normal tertiary</i> Ionization alone in the Ionization-chamber I.					
	TERTIARY RADIATORS.											
	Cr.	Fe.	Co.	Ni.	Cu.	Zn.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.
Cr	·076	·107	·109	·143	—	—	—	—	—	—	—	—
Fe	1·76	·076	·091	·118	·126	·124	1·68	—	—	—	—	—
Co	1·81	·42	·097	·118	·100	·106	1·73	·327	—	—	—	—
Ni	1·95	2·64	·466	·101	·084	·087	1·87	2·45	·366	—	—	—
Cu	1·92	2·64	3·00	·582	·093	·110	1·84	2·55	2·90	·472	—	—
Zn	2·02	2·88	3·37	3·53	·528	·115	1·94	2·79	3·27	3·42	·488	—
Ga?
Ge?
As	2·02	3·10	3·57	3·79	3·97	4·03	1·94	3·01	3·47	3·68	3·86	3·92
Se	1·86	2·87	3·11	3·47	3·61	3·86	1·78	2·78	3·01	3·36	3·50	3·75
...
...
...
...
...
...
Ag	—	—	—	—	1·35	1·40	—	—	—	—	1·24	1·29

However, it is found that when zinc is used as a tertiary radiator and the secondary beams from iron to zinc fall upon it successively, a small value R is obtained, even though none of the characteristic homogeneous radiation is excited by these secondary beams. The ionization in the chamber I in

these cases is due, partly to the scattering of the secondary radiation, partly to the production of tertiary radiation by the small percentage of scattered primary radiation in the secondary beam employed, and possibly in part to the production of a feeble type of very easily absorbed radiation much softer than the normal homogeneous radiation from zinc.

All evidence obtained up to the present points to the persistence of these residual effects even when the exciting secondary beam is sufficiently penetrating to produce the characteristic zinc radiation, and since in the calculation of k we only require a measure of the homogeneous tertiary radiation excited by the homogeneous secondary beam, these residual effects have been estimated and subtracted in each case.

The values of R finally obtained, which we may denote by R' , are given in Table VI.

The next step was to calculate in some one particular case an actual value of k from the data obtained by using a suitable pair of secondary and tertiary radiators. A flat polished plate of pure copper (3.04 cm. \times 3.03 cm.) was mounted on a fine aluminium stem and placed in the position R_2 . The ionization-chamber connected to the tertiary electroscope E_3 was removed and replaced by the electroscope E_2 . The distance from the centre of the radiator to the centre of the tissue-paper-covered window of E_2 was 4.7 cm., the area of the window being the same as in previous experiments. The plane of the radiator being vertical with the normal to its surface bisecting the angle between the axis of the secondary beam and the normal to the aperture at its centre, of the electroscope E_2 .

From the relative positions of the tertiary radiator and the window of E_2 (carefully determined) a value of ω was calculated and found to be approximately .222.

The ratio of the ionizations in the electroscopes E_2 and E_1 was then determined, allowance being made for the air-effect in the manner previously explained, and the mean of several readings, none of which differed by more than 2 per cent. from the mean, was found to be .115.

The tertiary radiator was then removed and the electroscope E_2 was placed with its window perpendicular to the axis of

the secondary beam, the centre of the window being in the position previously occupied by the centre of the tertiary radiator. Since the area of the window of E_2 was almost exactly equal in shape and size to the area of the tertiary radiator projected in a plane perpendicular to the axis of the secondary beam, the ionization will correspond to the mean intensity over the area of the tertiary radiator in the previous part of the experiment. The mean value of the ratio of the deflexions in E_2 and E_1 was now equal to 11.48.

Substituting the values of α , β , λ_1 and λ_2 from Table V., we find

$$k = \frac{.115}{11.48} \times \frac{60.7}{128.9} \times \frac{6.0}{9.2} \times \frac{4\pi}{.222} \times (1575 + 488) = 361$$

The value of λ_2 was increased by 3 per cent. to allow for the obliquity; the value of k was, however, affected by less than 1 per cent.

In the series of experiments previously mentioned, in the course of which a copper radiator was subjected to secondary beams from the group of metals Cr-Ag, we obtained values of R' with arsenic as radiator, and in these cases

$$k = \frac{R'}{\mu} \cdot \frac{\alpha}{\beta} \cdot \frac{A}{S'} \cdot \frac{4\pi}{\omega'} (\lambda_1 + \lambda_2),$$

where μ is the ratio of the sensibility of the electroscope E_3 to that of E_2 ; S' is the area of the tertiary radiator and ω' the mean solid angle subtended by the radiator R_2 at the window of the ionization-chamber I; and α , β , λ_1 , and λ_2 as defined above and corrected for the obliquity of the rays. This correction was determined experimentally.

Equating these values of k , we find

$$\frac{4\pi A}{\mu \omega' S'} = .093,$$

and we may therefore write

$$k = .093 \times R' \times \frac{\alpha}{\beta} \times (\lambda_1 + \lambda_2).$$

The value of k calculated from the data given in Tables V. and VI. are given below.

TABLE VII.

SECONDARY RADIATORS.	Values of k .						Values of $\frac{k}{\lambda'}$.					
	TERTIARY RADIATORS.											
	Cr.	Fe.	Co.	Ni.	Cu.	Zn.	Cr.	Fe.	Co.	Ni.	Cu.	Zn.
Iron	378						·176					
Cobalt	275	31·5					·154	·309				
Nickel	216	526	38·4				·145	·252	·287			
Copper	162	403	670	43·2			·127	·225	·305	·288		
Zinc	117·5	307	528	657	43·0		·109	·207	·282	·340	·358	
Arsenic ...	50·5	137	227	288	390	412	·0757	·150	·195	·233	·288	·327
Selenium ...	36·0	96·0	151·0	197	267	300	·0624	·120	·149	·187	·231	·278
Silver					5·35	5·30					·0282	·0313

NOTE.—In the case of each tertiary radiator, when the secondary beam is only just more penetrating than the radiation characteristic of the tertiary radiator, the value of k is small and cannot be determined nearly as accurately as the later values; nor can the value of λ' corresponding to the initial small value of k be determined with accuracy. Consequently the initial values of k are somewhat uncertain, and it is impossible to determine from them whether $\frac{k}{\lambda'}$ for any given tertiary radiator rises to a maximum, and then decreases in value, or whether it continuously decreases from the initial value.

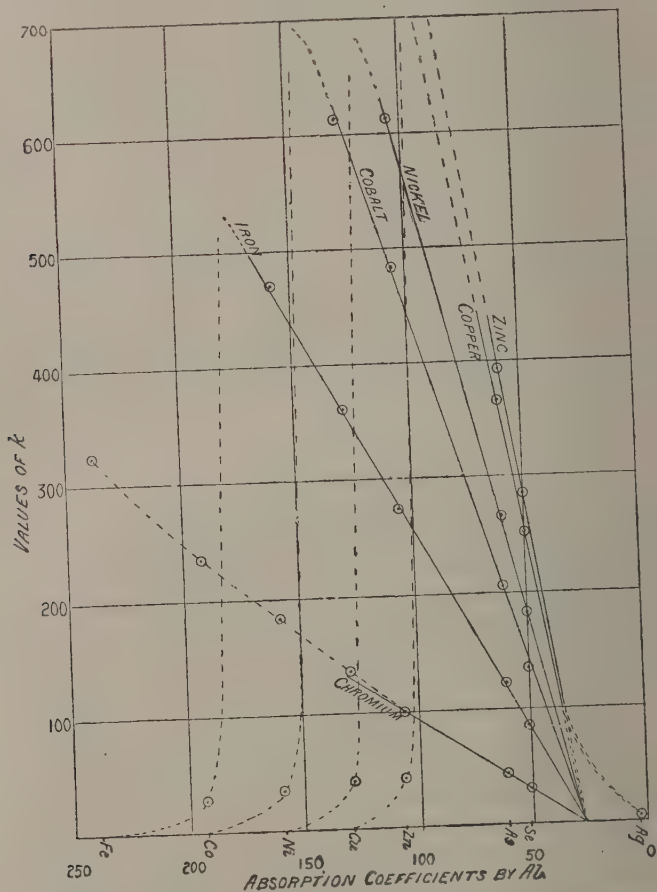
If the values of k for each tertiary radiator are plotted as ordinates against the absorption coefficients by aluminium of the exciting secondary beams as abscissæ, we obtain the graphs shown in fig. 4, p. 768.

If λ_2 be the value of the absorption coefficient by Al of the characteristic homogeneous radiation from any substance A and λ_1 that of the exciting radiation (also homogeneous), an examination of the graphs in fig. 4 leads to the following conclusions:—

(1) If λ_1 is equal to or greater than λ_2 , the intensity of the homogeneous radiation emitted by A is inappreciable.

(2) If λ_1 is slightly less than λ_2 the intensity is measurable but still small.

Fig. 4.



(3) For a further slight decrease in λ_1 a very rapid increase in the intensity of the radiation from A takes place, and for a value of λ_1 not greatly less than λ_2 the intensity reaches a maximum value.

(4) Beyond this point, over a considerable range, as λ_1 decreases the intensity decreases as a linear function of λ_1 . For this range the relationship may be represented by $k=a(\lambda_1-b)$, where a and b are constants. For the group of tertiary radiators employed, b has the same value for each member of the group, while a increases with the increase of the atomic weight of the tertiary radiator.

But the absorption by Al of the radiation from the group of metals Cr-Ag is proportional to the absorption by air of these radiations, as previously mentioned; and it has been shown that the intensity of the homogeneous radiation excited in a thin sheet of copper by a primary beam is proportional to the ionization produced in air by the same beam*: we may, therefore, over this range, write $k=a(ci-b)$, where i is a measure of the ionization produced in 1 c. c. of air by the exciting beam, and c the ratio of the absorption by 1 c.c. of air to the ionization produced in it in consequence of that absorption, a and b having the same values as before. (It will be seen from the Zn and Cu curves that there is a departure from this linear relationship for very penetrating beams.)

When a secondary homogeneous Röntgen beam falls normally upon a tertiary radiator, the amount of energy absorbed per sec. per unit area of its surface in a layer of thickness δx at a distance x below the surface is $I\lambda_1\delta x$, where I is the energy crossing unit area of the surface per second at the depth x below the surface and λ_1 is the absorption coefficient by the substance of the tertiary radiator of the secondary beam. If, however, we confine our attention to the absorption which is directly involved in the process of emission of tertiary rays, we must substitute λ' for λ_1 , where λ' is the increase in the value of the absorption coefficient, and the amount of energy so absorbed per second in this layer is $I\lambda'\delta x$. [The values of λ' are given in Table V.]

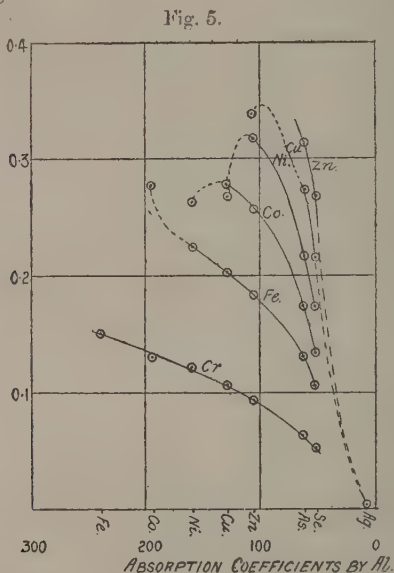
But the amount of energy emitted as tertiary radiation per second from this layer in consequence of such absorption is, as we have seen, $Ik\delta x$.

Therefore $\frac{k}{\lambda}$ is a measure of that fraction of this special

* Barkla & Sadler, Phil. Mag. Oct. 1908, pp. 550-584.

absorption of energy which is re-emitted as energy of tertiary radiation.

The values of $\frac{k}{\lambda'}$ are given in Table VII., and the curves obtained by plotting these values as ordinates against the values of λ_1 by Al of the exciting radiation as abscissæ are shown in fig. 5.



For any given tertiary radiator the value of $\frac{k}{\lambda'}$ attains to a maximum for a value of λ_1 in the neighbourhood of that which gives a maximum value of k . As the exciting radiation becomes more penetrating this fraction diminishes slowly at first and then more rapidly.

Taking the case of copper as a tertiary radiator for example, the maximum value of the fraction which corresponds to a value of λ_1 about 89 is nearly $2/5$, when λ_1 decreases to 51 the fraction has fallen to about $1/4$, and for a value of $\lambda_1 = 6.96$ to about $1/40$.

At the present stage of the inquiry the cause of this rapid decrease in the value of $\frac{k}{\lambda'}$ can merely be a matter for

conjecture, but it is hoped that the data obtained in subsequent experiments will throw more light upon the phenomena*.

It will be observed that in determining these relationships the atomic weight of nickel is nowhere assumed, but the results obtained all point to the conclusion that the behaviour of nickel, whether as radiator or absorber of X-rays, is such that we should expect its atomic weight to lie between those of cobalt and copper.

The evidence for this, obtained during the course of these experiments, may be briefly summarized as follows:—

(a) The secondary homogeneous radiation from cobalt excites *no* homogeneous tertiary radiation in nickel.

(b) The secondary homogeneous radiation from nickel excites *no* homogeneous tertiary radiation in nickel.

(c) The secondary homogeneous radiation from nickel *does* excite homogeneous tertiary radiation in cobalt, and to an extent to be expected for the radiation from a substance of atomic weight about midway between those of cobalt and copper.

But it has been shown for the group of metals Cr-Ag that

(1) The penetrating power of the radiation characteristic of any member of the group increases with its atomic weight †.

(2) The radiation characteristic of a substance is only excited by a more penetrating radiation ‡.

If the atomic weight of nickel were greater than that of cobalt, then by (1) we should expect to find its characteristic radiation to be more penetrating than that of cobalt, and consequently by (2) that the radiation from nickel *should* excite in cobalt its characteristic radiation, while the radiation

* *Note.* Experiments now in progress indicate that when the exciting beam is more penetrating than the radiation characteristic of the tertiary radiator, part of the energy absorbed reappears as an easily absorbed corpuscular radiation, and this effect becomes more pronounced as the exciting beam becomes more penetrating.

† Barkla & Sadler, *Phil. Mag.* Sept. 1907, pp. 812-828; also *Phil. Mag.* May 1909.

‡ Barkla & Sadler, *Phil. Mag.* Oct. 1908; also present paper.

from cobalt should *not* excite the radiation characteristic of nickel. If, on the other hand, the atomic weight of nickel were less than that of cobalt, then the results would be the reverse of the former.

It has been suggested that the anomalous behaviour of nickel may be due to oxidation of the cobalt or the nickel. The author has found, however, that if iron be used as a tertiary radiator, and the secondary radiations from the group of metals Cr-Ag successively fall upon the iron in the manner described earlier in the paper, the results obtained are exactly similar whether the radiator is coated with a layer of rust or brightly polished, the relative values of k being the same in the two cases.

The result (*b*) would not have been obtained if the nickel experimented with had contained an impurity which emitted a characteristic type of radiation "harder" or "softer" than that characteristic of nickel. For the harder component of the secondary beam (whether due to the nickel or the impurity) would have excited the softer component radiation in the tertiary radiator, and this would have been detected if the impurity had been present to any appreciable extent.

Summary of Results.

(1) The characteristic Röntgen radiation from a substance is found to be identical in character whether it is emitted as a secondary or a tertiary radiation. In each case (*a*) the radiation is homogeneous, (*b*) the absorption coefficients of the radiation by other substances are the same, and independent of the penetrating power of the exciting radiation by which these homogeneous beams are produced.

(2) Using the homogeneous secondary beams emitted by the members of the group of metals Cr-Ag, excited by suitable primary beams, the emission of tertiary radiation by the earlier members of the group Cr-Ag when excited by these secondary beams was found to be governed by the following laws:—(*a*) With a given substance as radiator, its characteristic radiation is only excited by those secondary beams which are more penetrating than the tertiary radiation characteristic of the substance. (*b*) When the secondary beam is only just more penetrating than the tertiary the

intensity of the latter is small, but as the secondary beam becomes more penetrating a very rapid increase in the intensity of the tertiary radiation to a maximum takes place. (c) As the secondary beam becomes more penetrating still, the intensity of the tertiary radiation decays as a linear function of the ionization produced in a given volume of air by the secondary beam.

(3) It has been found that when the secondary homogeneous beams from the group of metals Cr-Ag are absorbed by thin sheets of metals from the same group, a big increase in the absorption takes place when the secondary beams become more penetrating than the radiation characteristic of the absorber, this increase in the absorption being intimately connected with the emission of tertiary radiation by the absorber in these circumstances (see paper on "The Absorption of X-Rays," Barkla & Sadler, Phil. Mag. May 1909).

The fraction of this increase in the absorption of the energy of the secondary beam, which is re-emitted as tertiary radiation, is not constant, but decreases as the secondary beam becomes more penetrating, slowly at first, and then more rapidly when a very penetrating secondary beam is used.

In conclusion I wish to thank Dr. Barkla for the interest he has shown throughout this research, and especially for his kindly criticism and advice during the writing of this paper.

George Holt Physics Laboratory,
University of Liverpool,
24th March, 1909.

DISCUSSION.

Prof. BRAGG congratulated the Author on his interesting experiments, and said he could not see any satisfactory explanation of them on the pulse theory.

LII. *On a Bifilar Vibration Galvanometer.*

By W. DUDDSELL, Fellow of the Physical Society.

IN a paper read before this Society in May 1907, Mr. Albert Campbell drew attention to the great advantages of vibration galvanometers in the measurement of mutual inductances, and he further described a new type of vibration galvanometer which he had designed.

During the last few years the use of vibration galvanometers for the measurement of capacities, self and mutual inductions has greatly extended, and in the near future they may be expected to supersede the telephone and possibly the secohmmeter, for these purposes. As there are very few published data about the sensibility, etc., of these instruments I think it may be of interest to describe what I believe to be a new type and a series of tests made on it.

Vibration galvanometers like ordinary galvanometers may be broadly divided into two classes,—those in which the moving part consists of a piece of iron or steel and the current to be measured flows round fixed coils as in the case of the Thomson galvanometer,—those in which the current to be measured flows round a moving coil placed in a fixed magnetic field on the siphon recorder principle.

The vibration galvanometers of Max Wien and Ruben belong to the first class, while Mr. Campbell's moving coil vibration galvanometer belongs to the second, and so does the new bifilar instrument described below.

Generally speaking, when one requires to build a sensitive instrument having a short periodic time it is necessary to reduce as far as possible the mass of the moving parts in order to combine high sensibility with short period.

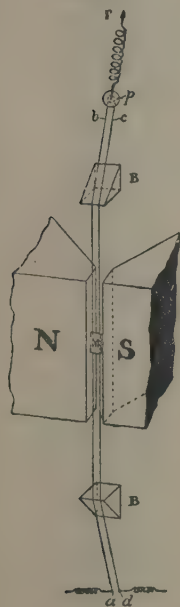
Further, in the case of a vibration galvanometer, it is also necessary to keep the damping forces as small as possible, as the sensibility to alternating currents depends very greatly on the magnification one can obtain by bringing the instrument into tune or resonance with the alternating current to be measured. These considerations have led me to construct a vibration galvanometer in which the mass of the moving

* Read May 14, 1909.

parts is reduced to a minimum, the moving coil being reduced to the two wires forming its two sides, similar to a bifilar oscillograph, but with this difference:—Whereas the bifilar oscillograph is designed so as to make the damping aperiodic, the bifilar vibration galvanometer is designed so as to keep the damping as small as possible.

The design of the instrument* is shown in fig. 1, in which

Fig. 1.



a, b, c, d, is a fine bronze wire passing over a pulley *p*, and stretched tight by means of a spring, the tension on the spring being capable of variation by a milled head. The wires carry a mirror *M* in the centre and are placed in a strong magnetic field between the poles *N* and *S* of a magnet. The wires pass over two bridge pieces *B, B*, which limit the length of the wires which is free to vibrate. These two bridge pieces can be moved nearer together or further apart by means of a right and left handed screw as required. The current to be measured passes up one wire and down the other, causing one wire to tend to move forward and the other back in the magnetic field and so tilts the mirror *M* through a small angle.

The periodic time of the wires depends on their mass, length, and tension, as well as upon the moment of inertia of the mirror. In a completed instrument the moment of inertia of the mirror and the

mass of the wires are fixed, but their length and tension can be altered in order to adjust the periodic time. Fig. 2 shows the relationship between the free length of the wires and the frequency of the free vibrations of the instrument for different tensions, and fig. 3 gives the relationship between the frequency and the tension for a series of different lengths. It will be seen from the curves that the total range of frequency obtainable with the instrument is very large, namely,

* Messrs. Nalder Bros. are manufacturing the instruments.

from about 90~ per second up to 1900, though the wires are rather too loose below 100~ per second.

Fig. 2.

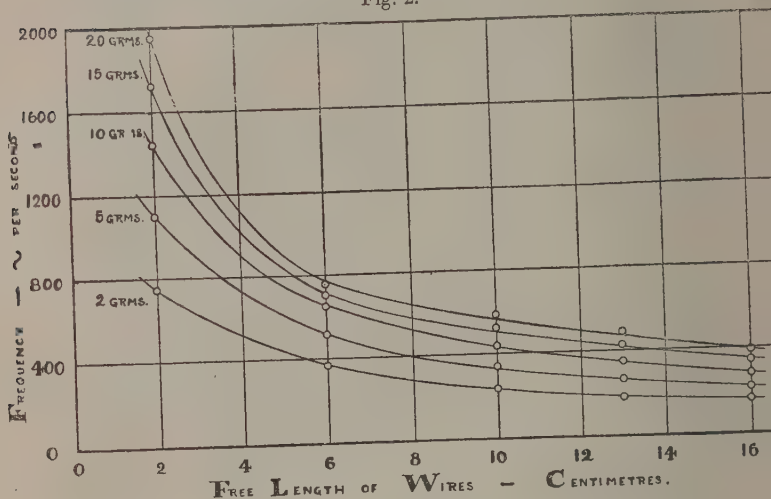
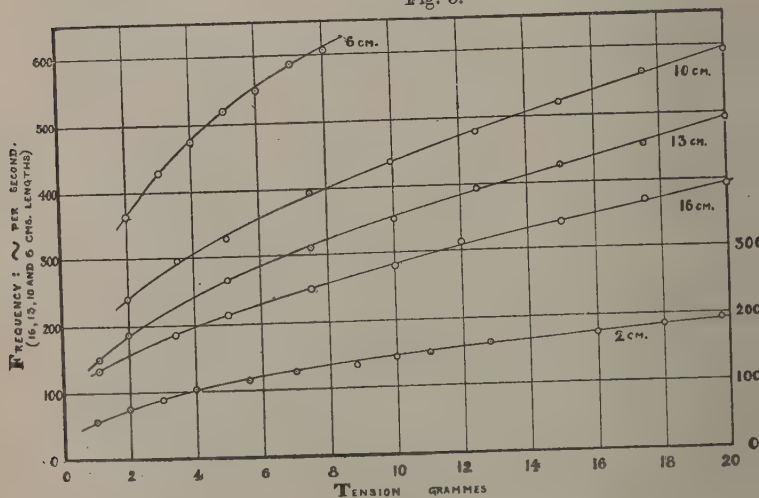


Fig. 3.



As the damping in this instrument is very small the resonance is very sharp. Figs. 4 and 5 show the amplitude

Fig. 4.

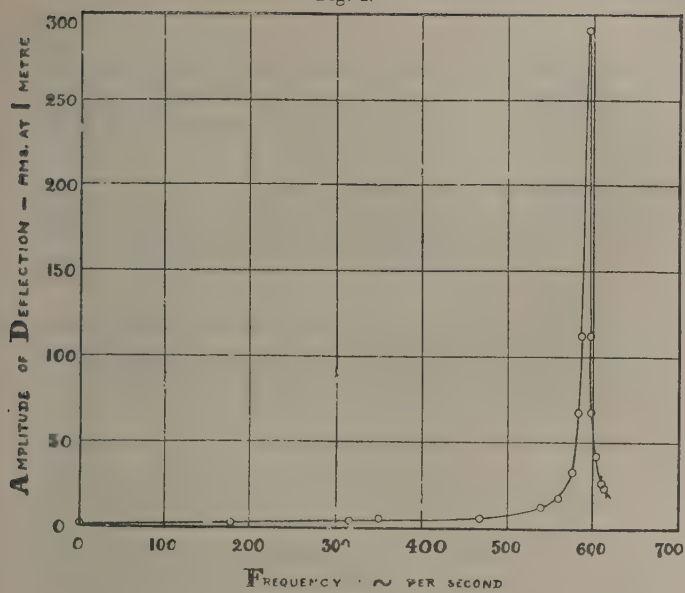
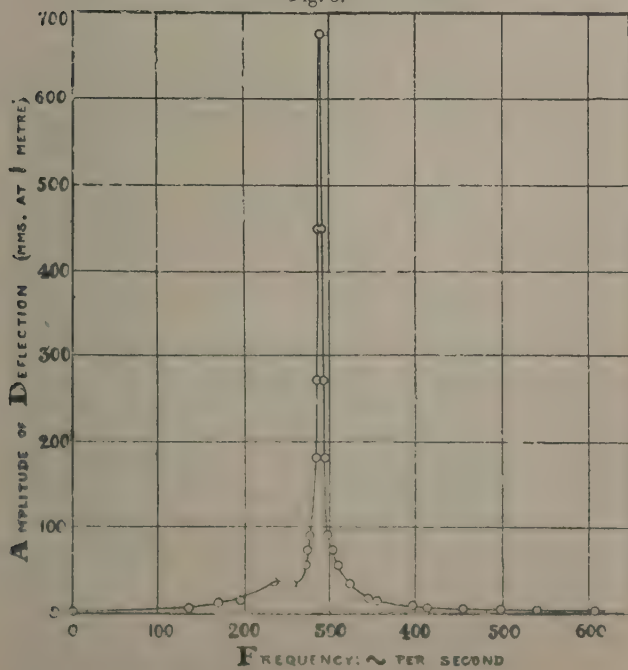


Fig. 5.



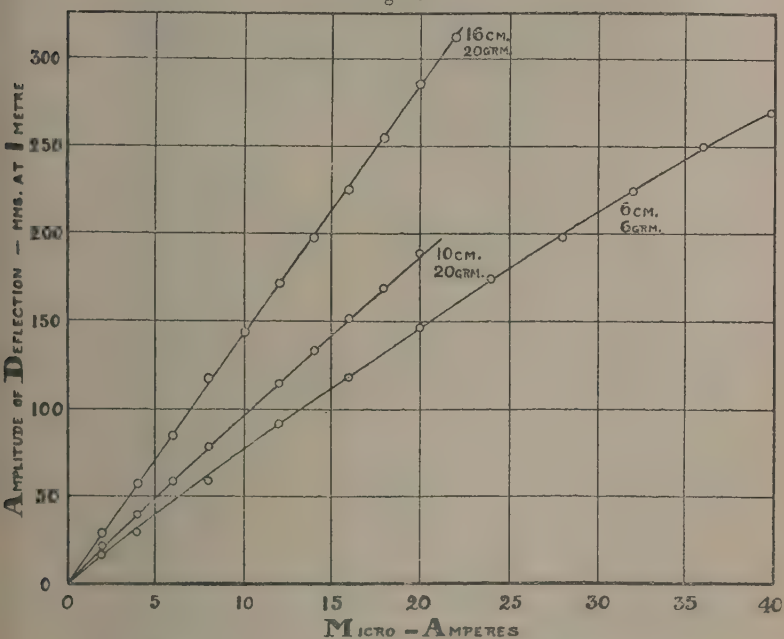
of the deflexion where an alternating current, having a constant R.M.S. value, is passed through the instrument, the frequency of the alternating current being varied. In fig. 4 the instrument was set so as to be in resonance at a frequency of 595~ per second, and in fig. 5 at a frequency of 290~ per second. The small irregularity in the curve fig. 5 at a frequency of 240~ per second was not an error of observation, but was, I think, due to the instrument resonating one of the higher harmonics of the wave form of the alternator.

To measure the sensibility of the instrument the following method was used:—A current generally 0.1 ampere from a small high-frequency alternator, having a very nearly sinusoidal wave-form, was passed through a small non-inductive resistance and an accurate dynamometer. The vibration galvanometer in series with a high non-inductive resistance was connected as a shunt to the terminals of the small resistance in the main circuit. The current of 0.1 ampere flowing through the main resistance produced a small known difference of potential applied to the galvanometer circuit from which the current through the galvanometer could be calculated. The vibration galvanometer it must be remembered, has when in operation a back E.M.F., so that it is very important to keep the resistance in series with the instrument very high to prevent the back E.M.F. from falsifying the calculation; 25,000 ohms was used for most of the tests. This method of testing the sensibility of the vibration galvanometer really calibrates the vibration galvanometer in terms of a standard dynamometer and known resistances. The vibration galvanometer is practically insensitive to anything except the fundamental of the wave-form of the alternating current for which it is tuned, whereas the dynamometer reads the mean squared current which is equal to the sum of the squares of the amplitude of all the harmonics including the fundamental. The two instruments are therefore not strictly comparable unless the amplitude of all the upper harmonics is zero, that is to say, that the current used has a pure sine wave-form.

With an instrument giving a very sharp resonance there is some difficulty in determining the exact value of the

maximum amplitude, a fact which prevents such consistent results being obtained as would otherwise be the case. When the instrument is tuned so as to be in resonance with the alternating current to be measured the amplitude of the vibration is practically proportional to the R.M.S. current, as is shown by the tests recorded in fig. 6, hence it is permissible to quote the sensibility in millimetres of amplitude at a metre per R.M.S. microampere. I have therefore adopted this method.

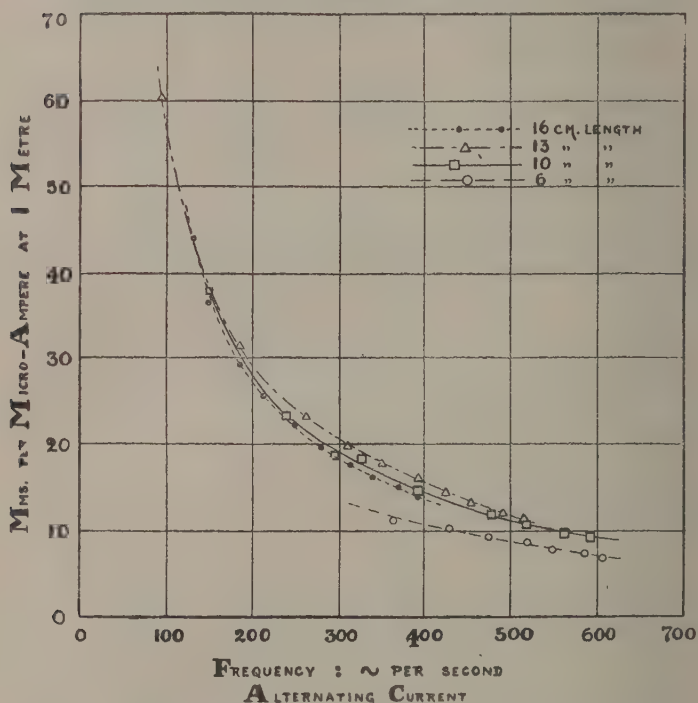
Fig. 6.



In fig. 7 (p. 780) the relationship between the sensibility of the instrument in millimetres of amplitude per R.M.S. microampere, and the frequency of the alternating current is given. The different curves refer to different free lengths of the wires. One interesting point of this figure is that where it is possible to tune this vibration galvanometer to a given frequency, using various combinations of length and tension,

the sensibility so obtained is very nearly the same as long as the wires are longer than the pole pieces. The practical result of this is that if at any given length of wire or tension, one can tune the instrument to suit the frequency of the alternating current, then no further adjustments need be made in

Fig. 7.



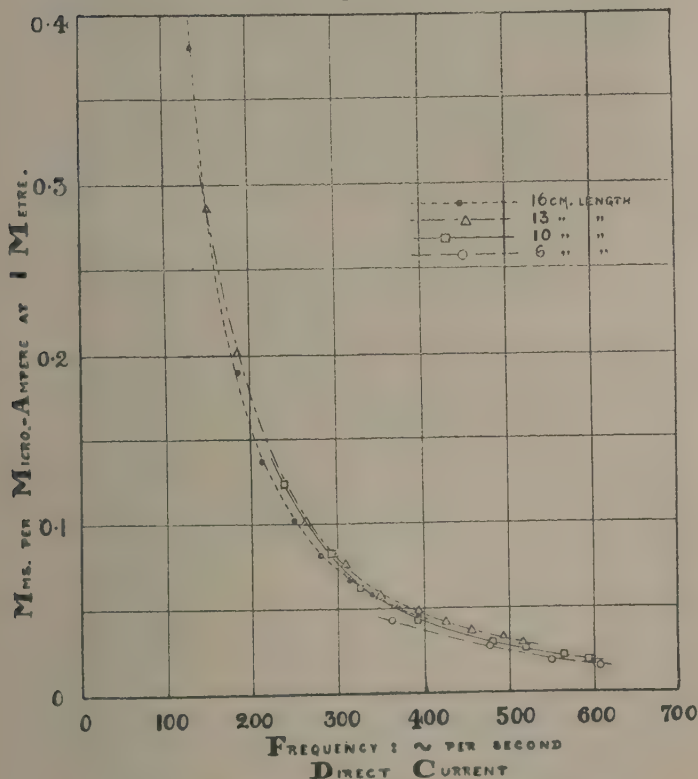
the hope of finding a better combination of length and tension for the purpose.

With each adjustment of the instrument in fig. 7 to suit the various frequencies, the sensibility of the instrument was tested with direct current as an ordinary galvanometer (fig. 8).

It will be noted that the sensibility to alternating current decreases very nearly inversely as the frequency for which

the instrument is adjusted, whereas for direct current the sensibility decreases approximately inversely as the square of the frequency for which the instrument is adjusted, which is what usually takes place with direct current galvanometers. By dividing corresponding ordinates in figs. 7 and 8 a new

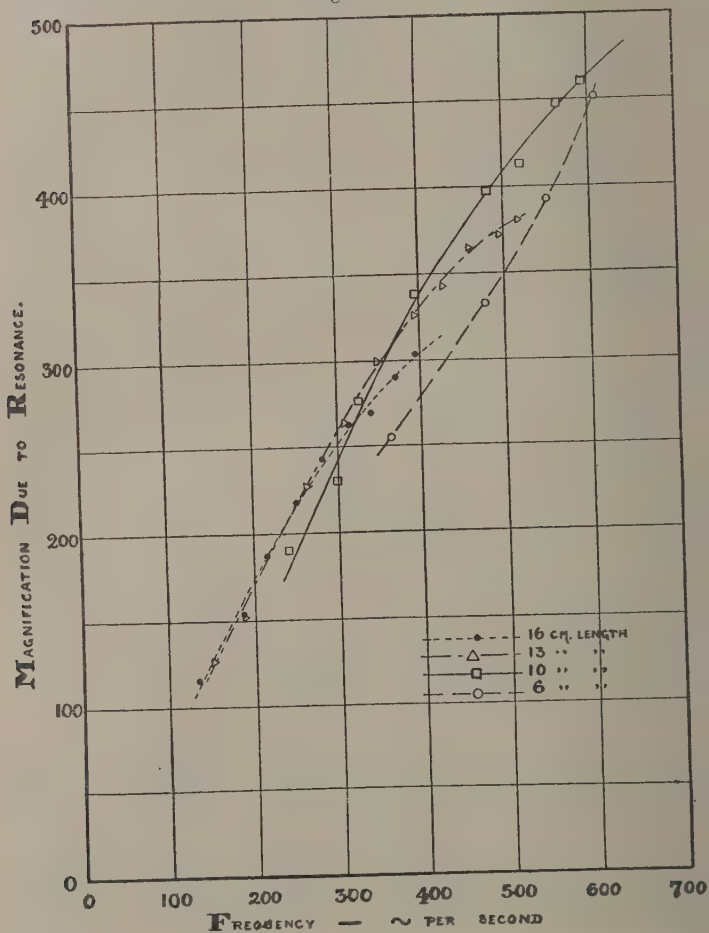
Fig. 8.



curve is obtained (fig. 9, p. 782), which I have called the magnification, which shows for a given adjustment of the instrument, how much more sensitive it is to an alternating current of the proper frequency than to a direct current. The highest value obtained in the curves is about 460. Had the vibration

galvanometer been aperiodic and of sufficiently short period to follow the alternating current then the alternating current sensibility would have been only $2\sqrt{2}=2.82$ times the direct current sensibility.

Fig. 9.



The practical applications of the vibration galvanometer nearly all involve using the instrument in some form of bridge or null method for determining when a small difference

of potential vanishes, that is to say the instrument is generally used as a sensitive detector for small alternating voltages. The resistance of the instrument used in these tests is 136 ohms, but owing to the back E.M.F. of the instrument its sensibility as a voltmeter must not be calculated on the basis of this figure. This resistance could be easily reduced by employing a more conducting material than hard phosphor bronze for the wires. There would be no objection to doing this except that the upper limit of the frequency to which the instrument could be tuned, without risk of breaking the wires, would be some somewhat reduced.

In view of the importance of the fact that the back E.M.F. reduces the sensibility of the instrument, I have made some tests to determine its value, using the same connexions as before :—

Let E = the R.M.S. value of the E.M.F. impressed on the circuit consisting of the vibration galvanometer and its added resistance ;

d = the corresponding amplitude of deflexion in millimetres ;

r = the total resistance of the circuit ;

e = the back E.M.F. (R.M.S. value per millimetre of deflexion) ;

c = the true value in microamperes of the current through the vibration galvanometer ;

k = the true sensibility in millimetres per micro-ampere ;

then $d = ck$.

If the mechanical friction of the instrument is small, then, when it is tuned to resonance, the motion of the wires will be nearly 90° out of phase with the force, *i. e.* the current ; and the E.M.F. induced in the wires due to their cutting the magnetic field will also be 90° out of phase with their motion, so that the back E.M.F. will be approximately at 180° to the current. The E.M.F. sending the current through the vibration galvanometer may therefore be approximately

taken as $E - ed$, and the current

$$c = \frac{E - ed}{r}, \text{ whence}$$

$$r + ek = k \frac{E}{d}.$$

If therefore $\frac{E}{d}$ be plotted against r the resistance of the vibration galvanometer circuit, the graphs obtained should be straight lines which should not pass through the zero but should cut the zero line at a point ek which gives the apparent resistance of the instrument due to its back E.M.F.

This test has been carried out and the graphs are given in fig. 10. From these graphs the following results were obtained :—

Free length in cm.	Frequency.	Intercept in ohms.	Back E.M.F. in microvolts per millimetre.
5	538	30	4.8
10	532	130	14.8
13	530	165	18.4
16	228	205	8.4

The very large amount that the apparent resistance of the instrument is increased by its back E.M.F., over 200 ohms in one case, shows how very important it is to try to keep this back E.M.F. as low as possible, and how little can be gained by reducing the real resistance of the wires themselves.

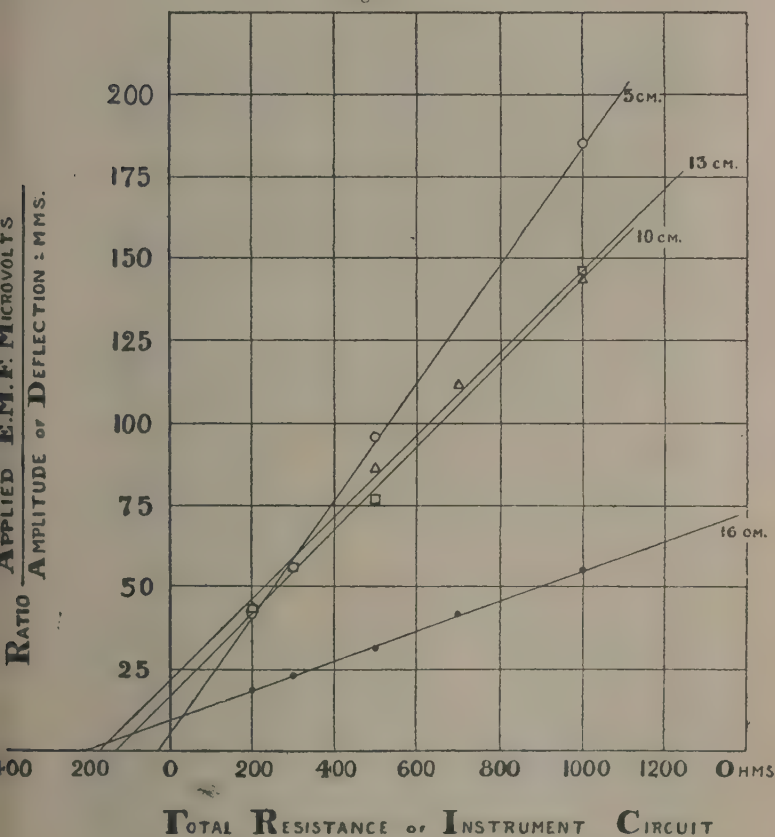
The fact that the back E.M.F. of the instrument is practically proportional to the deflexion and hence to the current through it, and so appears, as far as the outside circuit is concerned, like an addition to the resistance of the instrument, indicates that no arrangement of condensers is likely to improve matters as one might at first expect.

It will be seen that the intercept in the above table decreases for a given frequency with the length of the wires in use, and as the current sensibility does not change at all rapidly with the active length of the wires there is some

advantage in using the wires as short as possible when measuring small P.D.'s in a low resistance circuit.

I have not had an opportunity of testing the back E.M.F.

Fig. 10.



TOTAL RESISTANCE OF INSTRUMENT CIRCUIT

SCALE DISTANCE = 89 CMS. IN THIS CASE

of other vibration galvanometers. In the case of the moving coil vibration galvanometer I should expect that it would be considerably larger than in the bifilar owing to the coil having a number of turns. I trust that the discussion on the paper may bring out some information on this point.

All the sensibilities so far given for the vibration galvanometer correspond to the use of a permanent magnet for the field. The instrument has, however, been tried in an oscillograph electromagnet, and the current sensibility was found to be increased threefold.

This corresponds at a 100 frequency to a sensibility of not less than 160 millimetres per microampere; as a small fraction of a millimetre movement of the spot is noticeable we may reckon that at a 100 frequency with this instrument we can detect a current as low as $2 \text{ or } 3 \times 10^{-9}$ ampere.

The advantages of the bifilar galvanometer may be summarized as follows:—Simplicity—ease in tuning—wide range of frequency for which it can be tuned—high sensibility—negligible self-induction—comparatively small back E.M.F.

Its main defect is the small size of mirror that it is necessary to use on the instrument. With a carefully adjusted optical arrangement, and using a small 4 volt metal filament lamp, one can work with comfort at a scale distance of a metre in a room which is not too well lighted.

In conclusion, I wish to express my indebtedness to my assistant Mr. Neale, for the painstaking way in which he has made the experiments recorded in this paper.

DISCUSSION.

Mr. A. CAMPBELL said that thanks were due to the Author for an interesting account of the valuable and thorough investigation he had made of the behaviour of single-loop bifilar vibration galvanometers. Galvanometers of this type had been used at the Reichsanstalt for several years (*Zeitschr. für Instrumentenkunde*, May 1906) and had given good results with frequencies as high as 2500 ~ per second. He did not know what sensitivity was obtained with the German instruments, but there was no doubt Mr. Duddell's pattern of the same type was much more sensitive than the instrument of the moving coil type which he (Mr. Campbell) described in 1907. Blondel pointed out in 1900 that in an oscillograph there was no advantage in having more than one turn of wire, but this appeared to be only true when the mirror was very small. As a large mirror with plenty of light and good definition greatly improved the ease and accuracy of observation, he gave the preference to a moving coil of more than one turn, even though the single-loop instrument was highly recommended by Dr. Orlich. With a moderately large mirror one could obtain good definition of a dark line with a scale distance of 600 cms. Mr. Campbell mentioned that Wien's vibration galvanometer (*Ann. der Physik*, 1901) with very small

magnets and minute mirror gave 70 mms. at 1 metre distance for 1 microampere at 100 \sim per second and had a resistance of 200 ohms. Mr. Duddell's curves were most instructive. He had verified experimentally the law which he (Mr. Campbell) had stated from theoretical considerations, namely, that with a given bifilar suspension (tuned to resonance) the sensitivity varies inversely as the frequency. The figures given tended to show that the shorter bifilars gave reduced sensitivity. With much shorter bifilars he had found great loss of sensitivity. In the moving coil type with many turns the back E.M.F. was considerable and might even double the effective impedance of the instrument and lower the volt-sensitivity accordingly.

Dr. J. A. FLEMING asked the Author if he had tried his instrument with intermittent currents of the right frequency. If the galvanometer was sensitive to currents such as are obtained by rectifying trains of oscillations from condensers, it might be useful as an optical call in wireless telegraphic stations.

Dr. RUSSELL congratulated the Author on the notable advance he had made in the design of vibration galvanometers. He much appreciated the clear presentation and the accuracy of the experimental results. He asked whether variations of the barometric pressure had any appreciable effect on the sensitivity of the instrument, and suggested that variations in the humidity of the atmosphere might possibly have some effect. So far as he was aware, the Author was the first to point out the importance of the back electromotive force due to the vibrating wires cutting the magnetic field. He thought that the experimental results given would be a great help in formulating a more exact mathematical theory of this type of apparatus. He showed that to a first approximation the Author's results were in agreement with those deduced from the differential equation ordinarily given for the motion of the mirror. He asked whether the frequency of ordinary alternating-current supply circuits was sufficiently steady to avoid the necessity of constant tuning of the apparatus.

The CHAIRMAN asked the Author if the mirror was always situated in the middle of the vibrating fibres.

The AUTHOR, referring to Mr. Campbell's remarks, said considerations of space prevented him from working with scale distances of 6 metres. With regard to Wien's instrument it was easy to obtain high current sensibility but not high voltage sensibility. He had tried Prof. Fleming's suggestion, but his instrument was not sensitive enough for ordinary signals. With more uniform spark frequency it gave excellent results. In reply to Dr. Russell, he stated that the sensitivity of the galvanometer did vary slightly with the barometric pressure. In the ordinary differential equation the term giving the moment of the applied forces needed amending, and so also did the term for the damping couple. The complete equation was complex and there were difficulties in the way of getting an exact solution. The frequency of ordinary supply circuits was quite constant enough for accurate work. In reply to Dr. Chree, he stated that the mirror was always symmetrically situated on the suspension.

LIII. *On a Method of Testing Photographic Shutters.*

By A. CAMPBELL, B.A., and T. SMITH, B.A.*

(From the National Physical Laboratory.)

[Plate XXIX.]

I. *Principle of the Method.*

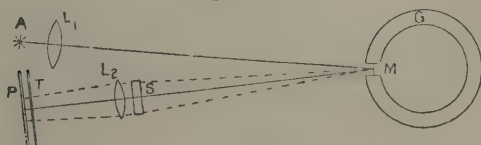
THE importance of having accurate methods of testing photographic shutters is well shown by the number and variety of the methods that have from time to time been proposed and used for this purpose. The method described herein is intended to provide a rapid test, while ensuring the maximum of accuracy. The total duration of exposure for low speeds (say longer than $\cdot 1$ sec.) is determined with an error not exceeding $\cdot 005$ sec., and for high speeds not exceeding $\cdot 0005$ sec. For exceptionally fast exposures, measurements can probably be made to within $\cdot 0001$ sec. Attention has been specially directed to the elimination of all calculations, and a permanent record is obtained of each test.

The essential principle of the method consists in photographing on a moving plate a narrow beam of light reflected from a mirror which is forced to make angular vibrations of known frequency about an axis parallel to the direction of motion of the plate. There is thus obtained on the plate a sine curve, and if the light on its way to the plate passes through the shutter, the length of the exposure can be found by counting the number of vibrations recorded on the plate. The method has been rendered practicable by the fact that a Vibration Galvanometer affords a suitable means of imparting the necessary oscillations to the mirror; with this instrument both the amplitude of the vibration and the frequency are under control, and these are points of great importance. Owing to these conditions not being satisfied, and for other reasons, a tuning fork is not nearly so suitable for imparting vibrations directly to the mirror. When it is desired to find

* Read May 14, 1909.

the total duration of the exposure only, the general arrangement is shown in fig. 1. Light from a point source A is reflected by the mirror M of the vibration galvanometer G, through the shutter S on to the photographic plate P. The mirror makes oscillations in a horizontal plane, and the

Fig. 1.



amplitude is such that the vibrating beam nearly fills the horizontal diameter of the shutter aperture. The plate falls vertically, and in doing so sets off the shutter. In practice, instead of a point source of light a Nernst lamp is used with its filament vertical, and in front of the plate is placed a narrow horizontal slit T to limit the width of the record on the plate. Lenses L_1 and L_2 are employed to regulate the intensity of the light and the amplitude of the vibration recorded as well as for focussing.

II. *Description of the Special Apparatus employed.*

(a) *Camera.*

A special camera has been constructed to hold the shutter S, the lens L_2 , and the slit T (fig. 1). It is provided with brass ways down which the plate-holder and plate slide, and an adjustable mechanism for setting off the shutter. A side elevation is shown in fig. 2, and a view of the back in fig. 3. A is the base carrying the whole apparatus, and into it is fixed a vertical pin B. C is a piece of wood in which is bored a hole through which B passes freely. C slides between two portions of the base D of the camera, and can be held in any position by tightening the nuts E. The front of the camera, with the shutter and lens L_2 , is supported by struts from C, and set so that the axis of the pin B, if produced, would pass through the centre of the shutter. By this means the whole vibrating beam of light will be able to pass through

the shutter though the whole camera is rotated through a few degrees about B. The back F of the camera is attached rigidly to the base D. The metal plate S carries a narrow horizontal slit T extending the whole width of the plate, and a window P is provided to facilitate focussing. G, G are

Fig. 2.

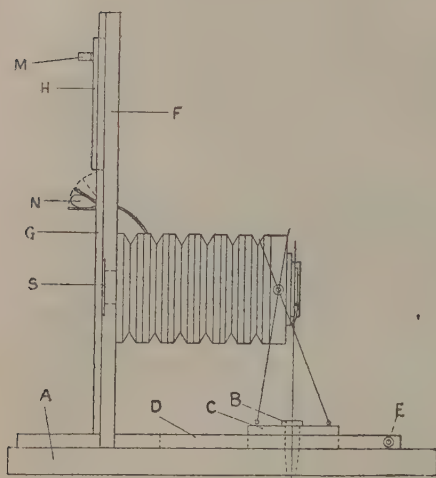
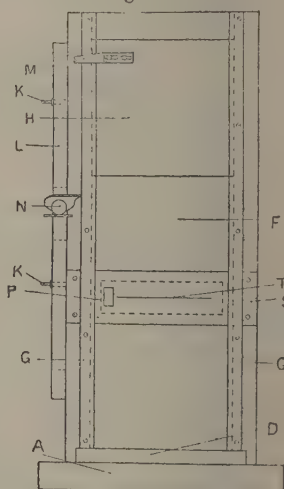


Fig. 3.



the grooved metal plates down which the plate-holder H slides. This holder carries a projection M which engages with the upper plate of a ball compressor N; when the holder falls this upper plate is forced down and the pneumatic bulb of the release is compressed. On the return upward journey of the holder the projection passes the compressor freely. The pneumatic release is carried on the moveable part L and may be fixed at any desired height by the butterfly screws K, K.

A number of records are taken side by side on one plate. This is secured by rotating the camera about B between each exposure. With this apparatus, ten or more records can be made side by side on a 5×4 in. plate. The base A has grooves cut in it to serve as a guide to the amount of rotation to be given to the camera. In making tests it is

possible to make on one plate ten records of the various speeds given by a shutter in one minute.

A view of the front of the camera is shown in fig. 9 (Pl. XXIX.).

(b) *Electrical Apparatus.*

The vibration galvanometer used is of the moving coil type (as already described by one of us*), and can be readily tuned to frequencies of 50 or 500~ per second. As its mirror is of fair size (50 to 80 sq. mm.), it gives with a Nernst lamp a sufficient amount of light to avoid "tailing off" in the records for quick exposures with rapid plates. The source of current may be an ordinary lighting circuit, the current through a lamp as resistance being made periodically intermittent by means of a wire interrupter, an electrically maintained tuning fork, or, for the higher frequencies, a microphone hummer†. The wire interrupter, which is merely an electrically maintained monochord, can be set to the desired frequency by the help of a tuning-fork; the maintained fork and the hummer give constant and known frequencies, and do not require setting. The pulsating current is led through a primary coil of 50 or 100 turns, and over this is placed a movable secondary coil connected directly to the vibration galvanometer, which is tuned to resonance with the source of current. The amplitude of its vibration can be brought to the desired amount by changing the position of the secondary coil relatively to the primary.

The distance from mirror to plate is usually about 100 cms.

III. *Determination of the Efficiency.*

For a more complete test the efficiency, in addition to the total duration of exposure, is determined. If τ denote the total duration of the exposure, T the equivalent exposure, a the area of the shutter aperture at the instant t , and A the maximum opening, we have the relation

$$AT = \int_0^{\tau} a dt; \text{ the efficiency} = \frac{T}{\tau} = \frac{\int_0^{\tau} a dt}{A\tau}.$$

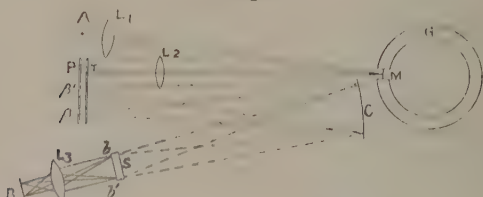
* Proc. Phys. Soc. vol. **xx**, and Phil. Mag. Oct. 1907.

† Proc. Roy. Soc. A, vol. **lxxviii**, p. 208 (1906).

Thus it is necessary to obtain a record of the area of the shutter opening at every instant of the exposure.

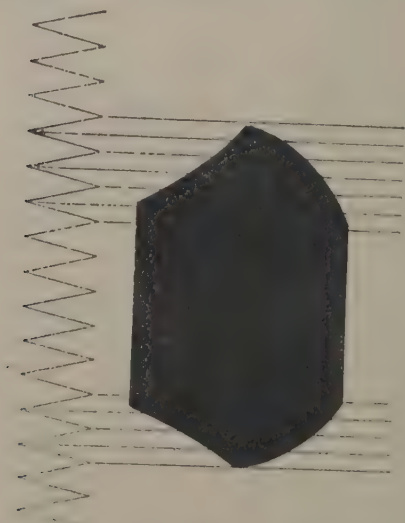
For the determination of the efficiency the method employed is essentially that proposed by Sir William Abney, who uses a siren to measure the time. We proceed as follows:—The

Fig. 4.



shutter is removed from the vibrating beam of light (see fig. 4) so that a continuous sine curve is recorded over the whole length of the plate, and this curve serves now simply

Fig. 5.



as a time record. A line source of light B is focussed by a lens L_3 on to a narrow horizontal slit bb' placed in a diametral plane of the shutter as close as possible to the shutter-leaves.

An image $\beta\beta'$ of this slit is formed by the concave mirror (C) on the photographic plate by the side of the vibrating beam of light. As the plate falls the shutter is opened as before, and a record is obtained giving at every instant of time the length of slit through which light was passed by the shutter leaves (fig. 5). Measurements are then taken of the area of shutter aperture corresponding to a number of lengths of the slit opening. Fig. 6 shows some stages in the opening of a particular type of shutter, the white line representing the position of the slit. From a pair of records such as shown in figs. 5 and 6 a curve is drawn showing at every instant of the exposure the area of the shutter aperture through which light can pass to the plate. It is now evidently a simple matter to find the amount of light cut off owing to the finite length of time taken by the shutter leaves to open and close, and hence to calculate the efficiency.

IV. *Examples of Records.*

A number of shutters of different types have been tested with the apparatus. Figs. 7 and 8 (Pl. XXIX.) are examples of the records obtained, the former at 50 and the latter at 500~ per second. Figs. 5 and 6 are copies of records obtained in testing the efficiency.

Fig. 6.



For general use the most convenient frequencies to work with are 50 and 500~ per second, with possibly 2000 for special work; or perhaps 100 and 1000~ per second would be suitable. The full advantages that the method offers can

only be secured by the use of round numbers, such as the above, for the frequencies of the oscillations.

In conclusion, we wish to thank Dr. Glazebrook for the kind interest he has taken in the working out of the method.

DISCUSSION.

MR. H. BECK expressed his interest in the method, which he said was the most satisfactory yet produced. It was difficult to say exactly what to call exposure. In photographing moving objects it was advisable to use as the effective time the time during which the central eight-tenths of the shutter was open.

MR. DUDDELL said the method was an ingenious one for determining the efficiencies of shutters. He suggested that by using an arc it might be possible to reduce the size of the mirror and thus work at higher frequencies. Instead of using a slit it might be better to use a short focus cylindrical lens.

MR. CAMPBELL, referring to Mr. Duddell's remarks, said that with a Nernst lamp it was possible to obtain high frequency curves showing very little tendency to tail off.

LIV. *Effect of Temperature on the Hysteresis Loss in Iron in a Rotating Field.* By W. P. FULLER, *B.Eng.*, and H. GRACE, *B.Eng.**

It was shown by Professor Baily † that the hysteresis losses due to a rotating field in iron reached a maximum value with an induction density (B) of about 16,000 C.G.S. units. With a value of B equal to 20,000 the hysteresis was approximately $\frac{1}{20}$ of the value. These results were confirmed by Messrs. Beattie & Clinker ‡. The experiments described below were undertaken in order to determine to what extent the results attained by Professor Baily were modified by variation in temperature.

In these experiments, the rotating field was produced by means of two phase-currents. Fig. 1 shows the arrangement, the two magnetizing coils carrying the two phase-currents being marked E, D. A is a slab of plaster 2 cms. thick and

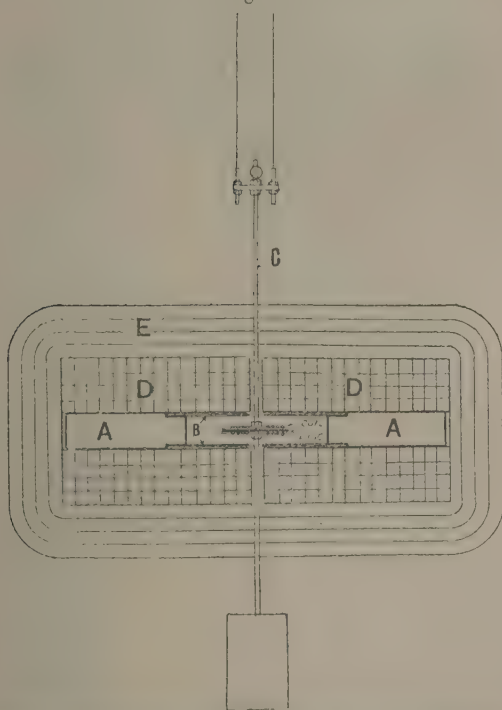
* Read May 14, 1909.

† Phil. Trans. 1896.

‡ 'Electrician,' 1896.

22 cms. square, having a circular hole $8\frac{1}{2}$ cms. diameter in the centre. At the top and bottom of the hole the heaters B are placed, and in the chamber so formed the iron specimen is suspended. Each heater was made by winding No. 40 s.w.g. nickel wire zigzag fashion over the surface of a circular piece of mica. Its resistance cold was 13 ohms, but

Fig. 1.



General arrangement of the magnetizing coils and specimen.

increased rapidly with temperature. With the two in series a current of 1.6 amps. at 180 volts produced a maximum temperature of 500°C . at a point close to the iron disk. The specimen used was a circular disk of iron 4 cms. diameter, .027 thick: it was attached by nuts to a brass spindle C, which had a weight attached to one end, and a *concave* mirror

at the other for indicating the motion of the specimen. The whole was supported by a bifilar suspension, the sensitiveness of which could be varied by altering the weight or by varying the distance apart of the supporting wires. The weight of the whole moving part was 285 grams.

The two magnetizing-coils D D, belonging to one phase, are each made by winding 14 turns of $7/14$ asbestos-covered cable. The two coils are arranged to slide over the plaster slab to within a short distance of each other, at the centre. The coils for the second phase, E E, are placed so as to produce a field at right angles to that of D D. They are of the same shape as D D, so that the two together produce a close approximation to a uniform rotating field at the centre of the coils where the specimen is placed. The large coil was found to produce a field 10 per cent. greater than that of the smaller for the same current, but with 25 amps in one and 27.5 in the other, the maximum values of the two fields were equal, each being about 250 c.g.s. units, and the two together producing a uniform rotating field of this magnitude. At a distance of 2 cms. from the centre of the coil the field produced was $1\frac{1}{2}$ per cent. less.

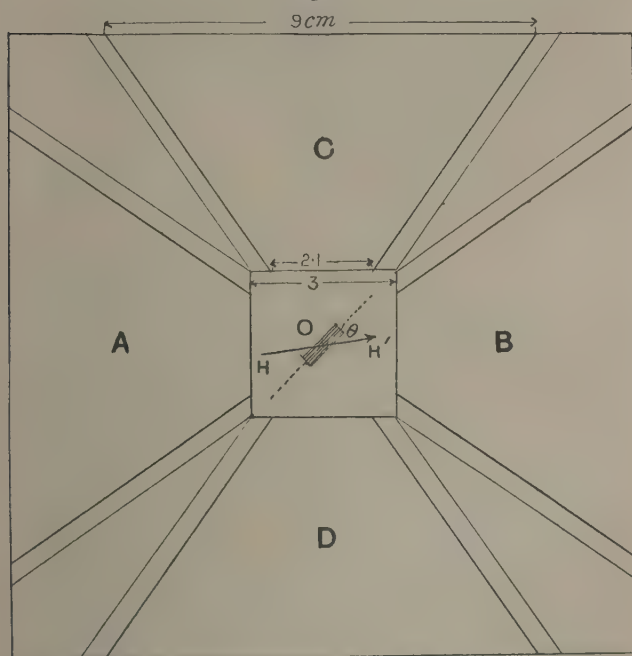
The phase relationship of the currents in the two phases was indicated by means of a wattmeter, the thick coil being in one circuit and the thin coil connected between the ends of a non-inductive resistance in the circuit of the other phase. A phase-difference of 90° between the currents in the two circuits was shown by a zero reading on the wattmeter. The phase-difference was kept within .3 per cent. of 90° by using a variable choking-coil in the primary of one of the transformers supplying current.

The numerous adjustments which have to be made are a disadvantage of the alternating current method compared with the rotating magnet method used by Baily and others; but the absence of rotating parts is a distinct advantage, and this fact would render the method very suitable for high-frequency experiments.

To measure the flux in the iron, a coil of 8 turns of bare wire was wound round it and insulated therefrom with mica. As the E.M.F. induced in the coil is alternating and of the order of .04 volt, a suitable galvanometer had to be con-

structed in order to measure it. The instrument constructed for the purpose is shown in fig. 2. A B C D are four coils of conical shape and wound with no. 28 wire. A and B were connected in series through a resistance to one phase, C and D to the other phase, of the machine supplying current to the magnetizing-coils of the test apparatus D E. With a current of .31 amp. a rotating field of about 20 C.G.S. units

Fig. 2.



was produced at the point O. A small coil, made up with 200 turns of no. 46 wire (140 ohms resistance) 5 cms. long and 1 cm. broad, is suspended in the central space between the coils, and is connected to the search-coil on the iron specimen. In the galvanometer-coil there are two E.M.F.'s, one, $E \cos pt$ (say) due to the flux in the iron specimen, and one due to the rotating field of the galvanometer itself. Referring to fig. 2, let H H' be the direction of the rotating

field of maximum value H when $E \cos pt$ is a maximum. The angle between it and the coil at a time t will be $\theta + pt$, and the component along the coil $H \cos \overline{pt + \theta}$. The flux normal to the coil is $H \sin \overline{pt + \theta}$, and the E.M.F. induced is proportional to

$$H p \cos \overline{pt + \theta} = E_1 \cos \overline{pt + \theta}.$$

If R be the resistance of the coil and the self-induction is negligible, the resultant current is equal to

$$\frac{1}{R} (E \cos pt + E_1 \cos \overline{pt + \theta}).$$

The torque at a time t will be proportional to

$$H \cos \overline{pt + \theta} \left\{ \frac{1}{R} (E \cos pt + E_1 \cos \overline{pt + \theta}) \right\}.$$

Hence the mean torque acting on the coil is proportional to

$$\frac{H}{R} \{E \cos \theta + E_1\}.$$

This deflecting torque therefore consists of two parts, one constant and one which depends on the position of the four magnetizing-coils A B C D. In the apparatus used these coils were fixed to a turntable, which could be rotated until the deflexion was a maximum. The deflecting torque would then be proportional to $\frac{H}{R} (E + E_1)$. Another way of using the instrument is to get E and E_1 in opposition, in which case the minimum deflexion is proportional to $(E_1 - E)$; and this is the better method because the total deflexion is smaller, and greater sensitiveness is obtained.

The instrument was calibrated by putting a resistance of 4000 ohms in series with the coil and applying a measured pressure of 1.95 R.M.S. volts to it. The deflexion, after deducting that obtained by shortcircuiting the coil through the 4000 ohms, equalled 22.4 centimetres, consequently one centimetre corresponds to a voltage of maximum value .00416 applied to the moving coil.

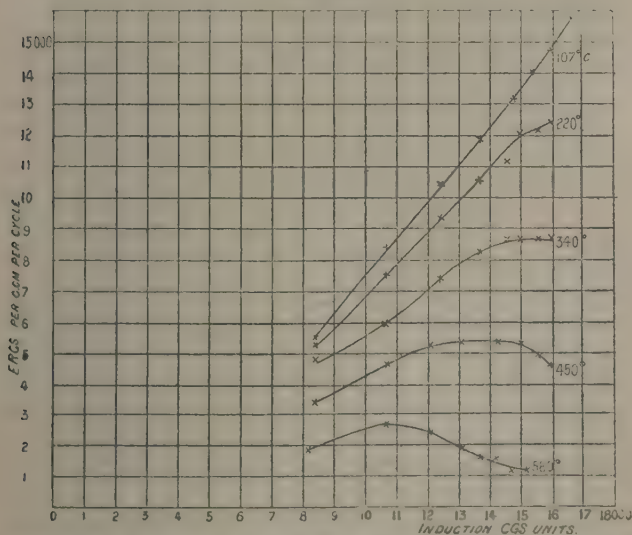
The induction density in the iron disk was readily calculated from the maximum voltage measured as above, the dimensions of the search-coil on the iron being known.

The disadvantage of this type of instrument is that the

constant part of the deflexion depends on the cube of the frequency, so that the speed of the machine must be very exactly regulated for a constant value. The E M.F. induced in the coil on the iron varies approximately as the frequency, and so the torque on the galvanometer due to this current depends on the square of the speed. It was found better in practice to have the two voltages E and E_1 in opposition, as mentioned above, so getting a deflexion proportionate to $E_1 - E$.

One other measurement required to be made, namely, that of temperature. This was effected by means of a Platinum-Platinum-Rhodium junction (placed close to the specimen) connected to a suitable galvanometer.

Fig. 3.

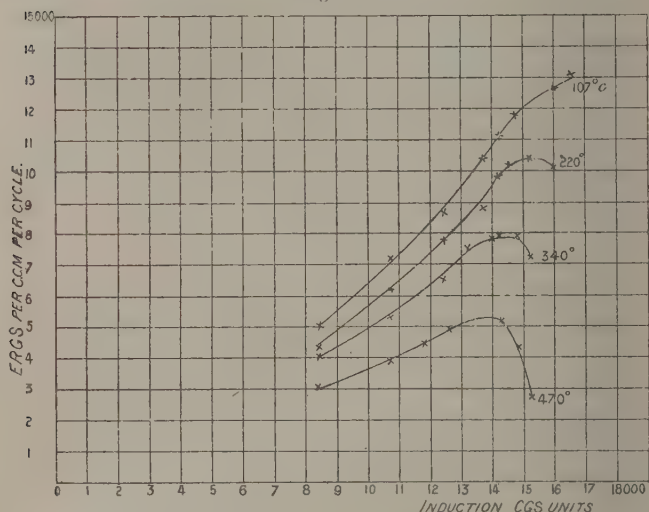


Curves showing relation between Hysteresis Loss and Induction at different Temperatures in a sample of armature iron from Messrs. Sankey.

The results obtained are shown in figs. 3 and 4. The temperature of 107° C. was the lowest at which it was found possible to make any measurements, on account of the heating effect of the magnetizing-coils. The results in fig. 4 were

obtained after the iron had been heated to 580° and cooled slowly. These experiments show that the effect of increasing the temperature of iron is to greatly reduce the hysteresis

Fig. 4.



loss at a given induction, and to cause the maximum to occur at a lower value of B . At a temperature of 580°C . the maximum hysteresis loss occurs at an induction density of 11,000 C.G.S. units instead of 16,000, while the maximum hysteresis loss is only 2600 as compared with a maximum of over 15,000 per c.c. per cycle at ordinary temperatures. The effect of heating to 580° , as shown in fig. 4, is most marked; although the actual maximum hysteresis loss is not greatly altered at the higher temperature, the reduction in the hysteresis loss at the lower temperature is considerable. The shape of the curve between ergs per cycle and B is quite different, the hysteresis loss falling off much more rapidly from the maximum.

The permeability of the iron is not greatly affected by changes in temperature within the limits of the experiment, as was shown by Morris*.

* Phil. Mag. Sept. 1897.

The above experiments were carried out at the Applied Electricity Laboratories of the University of Liverpool, and the authors thank Professor Marchant for his valuable advice and assistance.

DISCUSSION.

Dr. W. H. ECCLES remarked that the effects of previous thermal treatment, and of the nature of the material on the properties studied by the Authors, were so enormous that comparison of the present experiments with similar ones with alternating fields was difficult; but the Authors' curves seemed to show that the effect of rise of temperature on hysteresis loss was less for rotating fields than for alternating fields if the fields were below about 12 c.g.s. The curves showed also that the maximum values of hysteresis loss in rotating fields were reached at lower and lower flux densities the higher the temperature, although the permeability varied but little. This was of importance from the point of view of the molecular theory of magnetism: it showed that smaller forces sufficed to rotate the molecules within a magnetically stable group at high than at low temperatures. The Authors' experiments might be regarded as adding strong confirmation to the molecular theory of magnetism.

Prof. MARCHANT said the iron tested was obtained direct from the manufacturers and its previous thermal treatment was not known. A permeability curve had, however, been obtained. He expressed his interest in Dr. Eccles's remarks upon the bearing of the experiments on the molecular theory of magnetism.

Mr. A. CAMPBELL remarked that the methods and results described by the Authors were interesting, particularly the novel instrument employed. There is a source of considerable uncertainty, however, which does not appear to have been taken into account, namely that the total power spent in the iron disk includes eddy-current losses. For the thickness of sheet used, the eddy-current loss at 50 \sim per sec. and B maximum = 10,000 would be about 20 per cent. of the total amount measured when the induced voltage has sine wave-form. With other wave-forms the eddy-current loss depends on the square of the form factor and might easily rise to 40 per cent. of the whole, even although the source of current was a sine-wave alternator. Thus tests made without simultaneous measurement of the form factor are open to serious error. It is highly probable that the form factor would vary considerably along any one of the curves given in the paper. The eddy-current loss also would be very much reduced at the higher temperatures owing to the increased resistivity of the iron.

LIV. *The Arthur Wright Electrical Device for evaluating Formulae and solving Equations.* By ALEXANDER RUSSELL, M.A., D.Sc., and ARTHUR WRIGHT, M.I.E.E.*

TABLE OF CONTENTS.

- I. Introduction.
- II. Historical.
- III. The Slide Resistances.
- IV. Multiplication. The Index Line.
- V. The Contact-Fingers.
- VI. Addition and Subtraction.
- VII. The Use of the Arms of the Bridge.
- VIII. Cubic Equation.
- IX. Equations of Degrees higher than the Third.
- X. Imaginary Roots.
- XI. Solutions of Equations containing Miscellaneous Functions.
- XII. Solution of Transcendental Equations.
- XIII. Tracing Curves Electrically.
- XIV. Conclusion.

I. *Introduction.*

IN an ordinary algebraical expression consisting of several terms be computed by the slide-rule or by logarithmic tables, each term must be computed separately and the sum of the negative terms subtracted from the sum of the positive terms. If all these operations were done mechanically it would often save time and lessen the risk of error. In a few cases the form of the expression can be altered with advantage so as to adapt it to logarithmic computation, but there are many cases where the terms of the expression have to be evaluated separately, and in these cases it saves mental labour to do the operations mechanically.

The electrical device described below, which was invented by Mr. Arthur Wright, enables us to evaluate almost all mathematical expressions by simple mechanical and electrical operations. In the present model the accuracy is not high, being of the order of one per cent.; but this accuracy suffices in most engineering calculations. The apparatus, however, can be easily elaborated to give a far higher accuracy. The

* Read June 11, 1909.

main object of the authors in this paper is to explain the principles on which it works, and to point out a few of its many applications in practical calculations.

Although some of the operations described below—as, for example, the finding of the roots of a numerical algebraical equation of the n th degree—can easily be done by the expert mathematician, yet even in his case the apparatus can be a help, as it gives him at once the approximate values of the roots. In general, he can then find them to any required degree of accuracy by the methods given in the Theory of Equations.

II. *Historical.*

The history of calculating machines, and an excellent description of many of them, is given by M. d'Ocagne in his book *Le Calcul Simplifié* (1905). Valuable information is also given in Professor Henrici's article on "Mathematical Instruments" in the *Encyclopædia Britannica* (10th edition). We need only mention, therefore, the statical method* of solving equations due to Professor Vernon Boys, the Torrès logarithmic arithmophore†, which is a very elaborate piece of mechanism, and the extremely ingenious electrical method‡ due to M. F. Lucas, of finding the real and imaginary roots of an equation of the n th degree. The last method, however, is not rigorously accurate and would be very laborious to apply in practice.

III. *The Slide Resistances.*

In the electrical device described below the principle of the ordinary logarithmic slide-rule is combined with addition and subtraction, by utilising the laws according to which resistances combine in series or parallel. The products found by the slide-rule method are represented either by the resistances or by the reciprocals of the resistances of certain wires. In the latter case the currents through them can easily be added or subtracted by a Wheatstone Bridge Method. Hence the sums or differences of the products can be found.

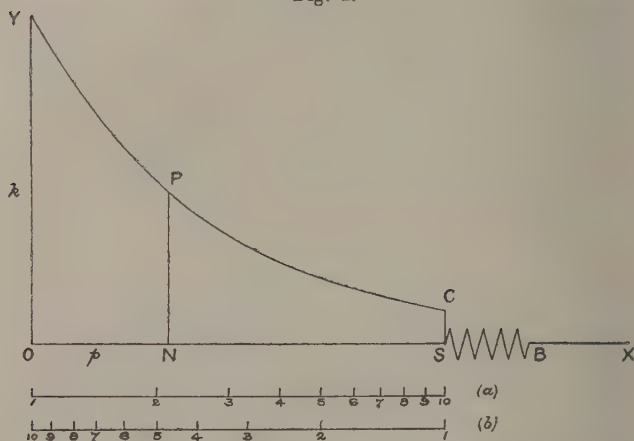
* Phil. Mag. vol. xxi. p. 241 (1886).

† M. d'Ocagne, *l. c. ante*, p. 123.

‡ *Comptes Rendus*, t. cvi. p. 1072 (1888).

The method of constructing the slide resistances (fig. 1) is as follows:—A template of thin insulating material is made

Fig. 1.



in the shape OSCY, shown in fig. 1. The equation to the curve YPC is

$$y = k 10^{-x/h},$$

where $OY = k$, $OS = h$, and $SC = k/10$.

This plate is wound uniformly with a hundred turns of No. 36 insulated manganin wire, the temperature coefficient of which is negligible, and the wires are practically parallel to OY. At points along OS the insulation of this wire is removed, and contact with the wires is made by the contact-finger described below. The scale OS is logarithmic. If p , for instance, be the reading at the point N, we have

$$ON = h \log p,$$

and thus $y = PN = k/p$.

The resistance of the wire between O and N will be approximately proportional to the area ONPY. This area is given by

$$\begin{aligned} \int_0^x y \, dx &= k \int_0^x 10^{-x/h} \, dx \\ &= \frac{hk}{\log_e 10} \left(1 - \frac{y}{k}\right). \end{aligned}$$

If, then, the wire on the frame have a resistance $9R/10$, and a coil SB of resistance $R/10$ be put in series with it, the resistance from N to B will equal $(y/k)R$, that is, R/p .

When the resistances are fixed on the device, the resistance between N and B is in circuit, and this always equals R/p , where p is the reading on the top scale [(a), fig. 1]. If we have n of these slide resistances in parallel, and their readings are $p_1, p_2, \dots p_n$, the sum of the currents will be

$$(E/R) (p_1 + p_2 + \dots + p_n),$$

where E is the applied potential-difference.

Instead of having the logarithmic scale placed as in (a), fig. 1, we may reverse it and place it as in (b). If p' be the value of ON read on this scale, we obviously have $p' = 10/p$; and hence the resistance between N and B is $(R/10)p'$. If the n resistances be now connected in series, their combined resistance will be

$$(R/10) (p_1' + p_2' + \dots + p_n');$$

and if E be the applied potential-difference, the current flowing through them will be

$$(10 E/R) / (p_1' + p_2' + \dots + p_n').$$

In some problems it is more convenient to use the scale placed as in (a), and in others it is more convenient to use it as in (b). In what follows, unless otherwise stated, we shall suppose that it is placed as in (a).

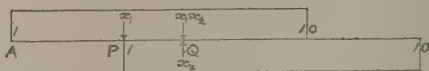
IV. *Multiplication. The Index Line.*

The method of performing multiplication is practically identical with that utilised in the ordinary slide-rule, which was originally designed by Seth Partridge*. In Partridge's

* 'The Description and Use of an Instrument called the Double Scale of Proportion.' London, 1671. Near the beginning of the book we read: "Here might have been expected a print of the rule, but in regard to its sliding it could not well be demonstrated: wherefore I thought good to advertise that this Scale and all other Mathematical Instruments are accurately made by Mr. Walter Hayes at the Cross-Daggers in More-Fields, next door to the Popes-Head-Tavern, London." Slide rules, therefore, were for sale in London 238 years ago.

instrument two logarithmic scales (fig. 2) slide with their edges in contact. If the reading on the top scale at P equals x_1 , and on the lower scale at Q equals x_2 , the reading on the top

Fig. 2.



scale at Q equals x_1x_2 . This follows because by the construction of the scales,

$$AP = h \log x_1, \quad PQ = h \log x_2,$$

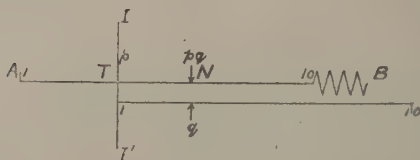
and thus

$$AQ = AP + PQ$$

$$= h \log x_1 + h \log x_2 = h \log x_1x_2.$$

In the Arthur Wright device the slide resistances are movable. If we wish to form the product pq , we move the slide so as to make the index-line II' (fig. 3) lie over

Fig. 3.



the reading p , and we then move the contact-finger until it makes contact at N, where TN , on the logarithmic scale, is q . The resistance NB will then equal R/pq .

V. The Contact Fingers.

The ordinary contact finger is a straight piece of metal wire ONP (fig. 4), which is capable of rotation about a point O fixed on a sliding bar parallel to and vertically over the index-line II' . This bar can move in the direction of its length. A pointer L attached to it moves along a fixed logarithmic scale KQ . The length of this scale is made equal to h , and so

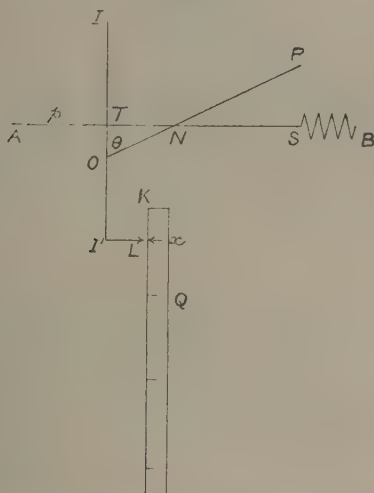
$$KL = h \log x = OT.$$

The contact finger OP can be adjusted so as to make any desired angle θ with the index-line II'. If the contact be made at N, we have

$$TN = OT \tan \theta = h \tan \theta \log x = h \log x^{\tan \theta};$$

and thus, if $AT=p$, the resistance from N to B will be

Fig. 4.



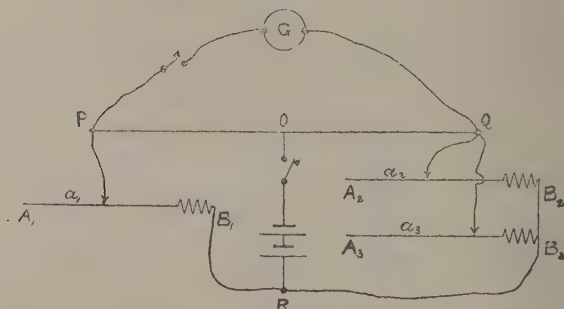
$R'(px^{\tan \theta})$. By giving various values to θ we can, with the help of a table of natural tangents, obtain all positive integral or fractional powers of x . If N lie between A and T, θ is negative, and so we obtain all the negative integral or fractional powers of x .

VI. Addition and Subtraction.

In performing addition and subtraction, the action of the apparatus can be understood from the following diagram. P and Q are the terminals of a wire bridge, of which O is the middle point (fig. 5). One pole of a battery of dry cells is connected with O and the other pole is connected with the B ends of the slide resistances, A_1B_1 , A_2B_2 , and A_3B_3 . A galvanometer connects P with Q. Let us suppose that

we have to find the value of $a_2 + a_3$. We connect Q with the contact-fingers of A_2B_2 and A_3B_3 respectively, and set the fingers at the marks a_2 and a_3 on their respective scales. We then join P with the contact-finger of A_1B_1 , and closing the keys we adjust the position of this finger until there is no

Fig. 5.



deflexion on the galvanometer. In this case let the reading on the slide A_1B_1 be a_1 . If E be the potential at R , and V be the potential at P or Q , then, since the currents in PO and QO are equal, we have

$$(E - V)/(R/a_1) = (E - V)/(R/a_2) + (E - V)/(R/a_3),$$

and thus

$$a_1 = a_2 + a_3.$$

To find $a_2 - a_3$, all we have to do is to change the connexion of the finger of A_3B_3 from Q to P . Proceeding as before, a_1 now gives us the value of $a_2 - a_3$.

When subtracting approximate values, it has to be remembered that the percentage error in the result may be large, especially when $a_2 - a_3$ is small compared with a_3 . This difficulty, which arises when subtracting the approximate values of nearly equal quantities, affects all approximate methods. In our case it can only be overcome by increasing the accuracy of the device.

VII. *The Use of the Arms of the Bridge.*

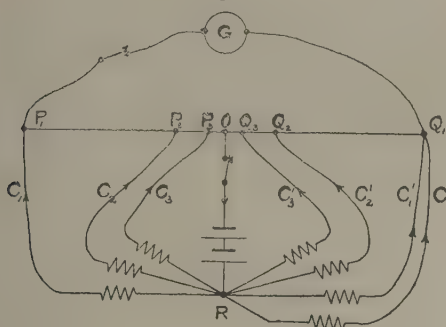
Terminals are connected at points P_1 , P_2 , P_3 , Q_3 , Q_2 , and Q_1 of the bridge wire (fig. 6). These points are chosen

so that if the resistance P_1Q_1 be $2r$, the resistances of OP_1 , OP_2 , and OP_3 are equal to r , $r/10$, and $r/100$, and also to the resistances of OQ_1 , OQ_2 , and OQ_3 respectively.

Let the magnitudes of the currents in RP_1 , RP_2 , RP_3 , RQ_3 , and RQ_2 be C_1 , C_2 , C_3 , C_3' , and C_2' respectively.

Let us also suppose that the currents in the two branches joining R and Q_1 are C and C_1' respectively.

Fig. 6.



The resistances in these circuits are the slide resistances described above. When there is no deflexion on the galvanometer joining P_1 and Q_1 , the potentials of these two points are equal, and so the differences of their potentials from that at O are equal, and thus we find that

$$C_1r + C_2(r/10) + C_3(r/100) \\ = Cr + C_1'r + C_2'(r/10) + C_3'(r/100).$$

Hence

$$C = C_1 + C_2/10 + C_3/100 - (C_1' + C_2'/10 + C_3'/100).$$

Let E be the potential of R , and V_1 the potential of P_1 and Q_1 (fig. 6). Then, if V_2, V_3, V_2', V_3' be the potentials of P_2, P_3, Q_2 , and Q_3 , and $p_1, p_2, p_3, q_1, q_2, q_3$, and x be the readings on the scales of the slide resistances in the branches RP_1, RP_2 , &c., we have

$$(E - V_1)x = (E - V_1)p_1 + (E - V_2)(p_2/10) + (E - V_3)(p_3/100) \\ - \{(E - V_1)q_1 + (E - V_2')(q_2/10) \\ + (E - V_3')(q_3/100)\};$$

and therefore

$$x = p_1 + p_2/10 + p_3/100 + \Delta_p - (q_1 + q_2/10 + q_3/100 + \Delta_q),$$

where

$$\Delta_p = \frac{V_1 - V_2}{E - V_1} \cdot \frac{p_2}{10} + \frac{V_1 - V_3}{E - V_1} \cdot \frac{p_3}{100},$$

and

$$\Delta_q = \frac{V_1 - V_2'}{E - V_1} \cdot \frac{q_2}{10} + \frac{V_1 - V_3'}{E - V_1} \cdot \frac{q_3}{100}.$$

The readings p_1 and q_1 on the scales cannot be less than 1, and the readings p_2 , p_3 , q_2 , and q_3 cannot be greater than 10. Hence, remembering that $(V_1 - V_2)/(E - V_1)$ cannot be greater than $(V_1 - V_3)/(E - V_1)$, we see that if $(V_1 - V_3)/(E - V_1)$ be equal to or less than $1/53$, Δ_p is not greater than the hundredth part of $p_1 + p_2/10 + p_3/100$. Similarly, if $(V_1 - V_3')/(E - V_1)$ be equal to or less than $1/53$, Δ_q is not greater than the hundredth part of $q_1 + q_2/10 + q_3/100$. We have therefore to arrange the relative values of the resistances of the slides and the bridge wire so that this may be true. As we have pointed out above, however, the inaccuracy in the value of x depends on the relative values of x and $p_1 + p_2/10 + p_3/100 + \Delta_p$. When these quantities are nearly equal, approximate methods of computation fail.

By Ohm's law, we see from fig. 6 that

$$\frac{V_1 - V_2}{\frac{9}{10}r} = \frac{E - V_1}{R_1}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{V_2 - V_3}{\frac{9}{100}r} = \frac{E - V_1}{R_1} + \frac{E - V_2}{R_2}, \quad . \quad . \quad . \quad (2)$$

where R_1 and R_2 are the resistances of the circuits RP_1 and RP_2 . Hence it readily follows that

$$\frac{V_1 - V_3}{E - V_1} = \frac{9}{10} \cdot \frac{r}{R_1} + \frac{9}{100} \left(\frac{r}{R_1} + \frac{r}{R_2} \right) + \frac{81}{1000} \cdot \frac{r^2}{R_1 R_2}.$$

If R be the resistance of a slide, the minimum possible value of both R_1 and R_2 is $R/10$. Hence the maximum possible value of $(V_1 - V_3)/(E - V)$ is

$$9 \frac{r}{R} + 1.8 \frac{r}{R} + 8.1 \left(\frac{r}{R} \right)^2,$$

and this is less than $1/53$ when $2r/R$ is not greater than $1/287$.

Similarly, if ρ , R_1' , and R_2' be the resistances of the slides, the readings on which are x , q_1 , and q_2 respectively, we see that

$$\frac{V_1 - V_3'}{E - V_1} = \frac{9}{10} \left(\frac{r}{\rho} + \frac{r}{R_1'} \right) + \frac{9}{100} \left(\frac{r}{\rho} + \frac{r}{R_1'} + \frac{r}{R_2'} \right) + \frac{81}{1000} \left(\frac{r}{\rho} + \frac{r}{R_1'} \right) \frac{r}{R_2'}.$$

In the very unlikely case when $\rho = R_1' = \frac{R}{10} = R_2'$, $(V_1 - V_3')/(E - V_1)$ has its maximum possible value, and in this case it will be less than $1/53$ if r/R be not greater than $1/1100$. Hence the desired accuracy can be attained by making the resistance R of each slide 550 times greater than the resistance of the bridge wire.

It is to be noticed that the effect of connecting a slide resistance with P_2 instead of P_1 is to divide its value by 10. Hence, if we were adding 8.8 to 0.75, we would put the contact-finger of one slide on 8.8 and connect it with P_1 , and the contact-finger of another slide on 7.5 and connect it with P_2 . Suppose also that the value of x was 123. In this case we could only obtain a balance when it was connected with Q_3 .

VIII. *Cubic Equation.*

Let us suppose that the cubic equation is

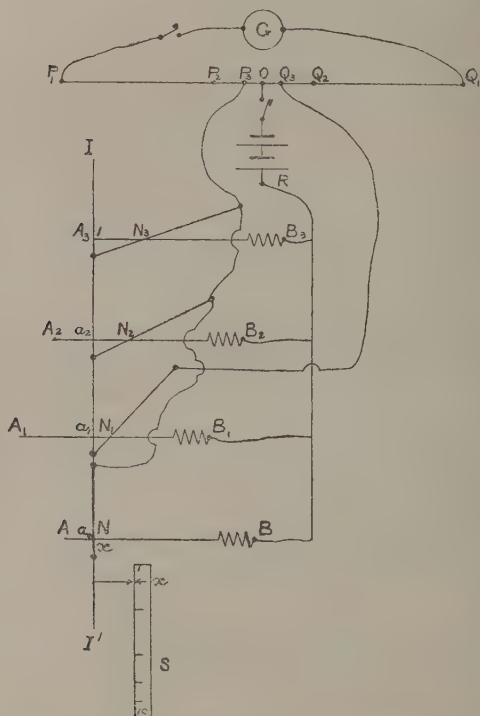
$$x^3 + a_2x^2 - a_1x + a_0 = 0,$$

where a_2 , a_1 , and a_0 are positive numbers.

The arrangement of the apparatus for solving this equation is shown in fig. 7, p. 812. The bridge-wire is the same as in fig. 6. We move the slides AB , A_1B_1 , A_2B_2 , and A_3B_3 so that the readings 1 , a_2 , a_1 , and a_0 on their logarithmic scales are on the index-line II' . The ends of the contact-fingers touching AB , A_2B_2 , and A_3B_3 are connected with P_3 and the finger touching A_1B_1 with Q_3 . The fingers through N , N_1 , N_2 , and N_3 make angles $\tan^{-1} 0$, $\tan^{-1} 1$, $\tan^{-1} 2$, and $\tan^{-1} 3$ with the index-line. The battery and galvanometer-keys being

closed, the bar to which the fingers are connected is gradually lowered. When the galvanometer deflexion is zero let us

Fig. 7.



suppose that x is the reading on the logarithmic scale S . In this case we have

$$\frac{E - V_3}{\frac{R}{x^3}} + \frac{E - V_3}{\frac{R}{a_2 x^2}} + \frac{E - V_3}{\frac{R}{a_0}} = \frac{E - V_3}{\frac{R}{a_1 x}},$$

and thus $x^3 + a_2 x^2 - a_1 x + a_0 = 0$. Hence x is one of the roots of this equation.

When a contact-finger, N_1 for instance, reaches the end of its slide $N_1 B_1$ will be $R/10$. If we now move the slide $A_1 B_1$ so so that A_1 is on the index-line, unclamp the contact-

finger and move it parallel to itself until N_1 is over A_1 and then reclamp it, the resistance of N_1B_1 will be R . If, then, we disconnect the finger N_1 from Q_3 and connect it with Q_2 the deflexion on the galvanometer will not be altered, and so we can continue to increase x . It will be seen that as x increases from 1 to 10 the contact-finger N_3 traverses A_3B_3 three times. The first time N_3 gets to the end of its scale it is connected with P_2 and moved back to A_3 . The second time it reaches the end it is connected with P_1 and again moved back.

The device enables us to find any real root of the equation, greater than 10^n and less than 10^{n+1} . All we have to do is to find the root of the equation

$$x^3 + (a_2/10^n)x^2 - (a_1/10^{2n})x + a_0/10^{3n} = 0,$$

lying between 1 and 10, and multiply the result by 10^n . By solving a similar subsidiary equation we can find the approximate values of the roots of the equation which are less than unity.

We may use Newton's rule to find more accurate values of the roots. If a , for instance, be an approximate value of a real root of $f(x)=0$, $a - f(a)/f'(a)$ gives usually a very much closer approximation x_1 . The failing case is when we have two roots very nearly equal to one another. The device always indicates when this occurs. If two roots are each equal to a , or if they are approximately equal to a , the galvanometer deflexion instead of passing to the other side of zero when x becomes greater than a returns to the same side.

In practice if $f''(a)$ is large when $f'(a)$ is zero the device is very sensitive, but when $f''(a)$ is small and in general, therefore, when the roots are equal the device is not sensitive.

To find approximate values of the negative roots of the original equation we find by the device the roots of

$$x^3 - a_2x^2 - a_1x - a_0 = 0.$$

The imaginary roots* are most readily found by first finding as accurate a value x_1 as possible of the real root. We then divide the cubic expression by $x - x_1$ and equate the dividend to zero. The roots of this quadratic equation give the approximate values required.

* Cf. C. P. Steinmetz, 'Transient Electric Phenomena,' p. 136.

IX. *Equations of Degrees higher than the Third.*

The device can be usefully employed to find the values of the roots of equations higher than the third with sufficient accuracy for practical work. When calculating, for example, the requisite resistances for the motor controller of an electric car, the following equation * has to be solved

$$x^{n/(n-1)} + a_1x - a_0 = 0,$$

where n is the number of steps in the rheostat.

To solve this equation three only of the slides shown in fig. 7 are required. AB is connected with Q_3 , and A_1B_1 and A_2B_2 are connected with P_3 . The finger making contact with A_2B_2 is inclined at an angle $\tan^{-1}n/(n-1)$ with II' , and the fingers on A_1B_1 and AB are inclined at angles of 45° and 0° respectively. The value of x is then increased until a balance is obtained.

When solving equations containing large numbers of terms it is sometimes convenient to divide the equation by a suitable power of x . In this case the fingers making contact with slide resistances representing terms containing negative powers of x are inclined to the left of II' (fig. 8).

Let us consider, for example, a sextic equation. As shown above, we alter it so that the required root or roots are multiples or submultiples of the equation

$$x^6 - a_5x^5 + a_4x^4 + a_3x^3 - a_2x^2 - a_1x + a_0 = 0,$$

which lie between 1 and 10.

Dividing this equation by x^3 we get

$$x^3 - a_5x^2 + a_4x + a_3 - a_2x^{-1} - a_1x^{-2} + a_0x^{-3} = 0.$$

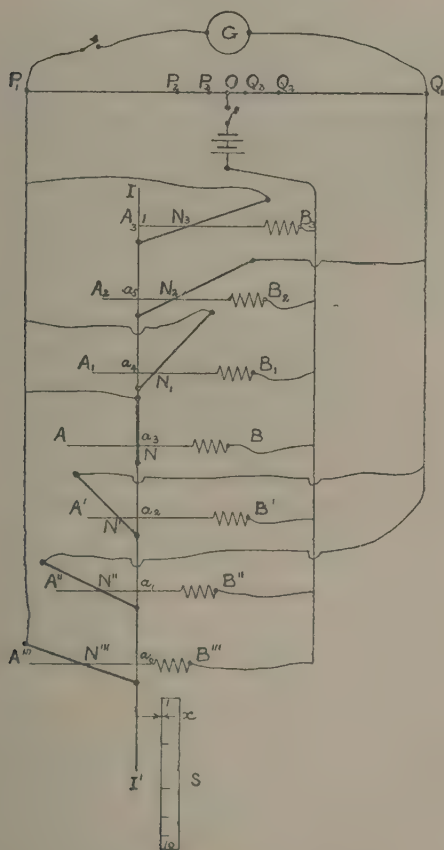
In general, seven slide resistances will be required, but if when x is put equal to unity any term is less than the hundredth part of the sum of the terms of the same sign preceding it, that term can be neglected. The slide resistances are first moved until the readings on the index-line are a_0, a_1, a_2, \dots &c., as shown in fig. 8. The contact-fingers are next turned round to the requisite angles $\tan^{-1}3, \tan^{-1}2, \dots \tan^{-1}-3$, respectively. The ends of the contact-fingers are

* E. Wilson, 'Electrical Traction,' vol. i. p. 42.

then connected with suitable points on the arms of the bridge.

At first sight it might be thought that two extra pairs of contact-points P_4, Q_4 and P_5, Q_5 would be required in the

Fig. 8.



bridge arms. If a one per cent. inaccuracy, however, is permissible this is not necessary. To illustrate this point, and also the method of using the device in this case, let us

consider the numerical equation

$$x^3 - 5.7x^2 + 0.12x + 8.6 - 0.04x^{-2} + 1.3x^{-3} = 0.$$

Putting x equal to unity we see that the term 0.04 can be neglected compared with 5.7 , only five slide resistances therefore are required. We connect A_3B_3 with P_2 , A_2B_2 with Q_2 , A_1B_1 with P_3 and make the reading on it 1.2 , AB with P_2 and $A'''B'''$ with P_2 . The finger on the slide $A'''B'''$ gets to the end A''' of its scale first. We then disconnect it from P_2 and connect it with P_3 . We also alter the reading on the slide resistance $A'''B'''$ to 10 . When the fingers on the resistances A_2B_2 and A_3B_3 get to the ends of their scales we connect them with P_1 and Q_1 respectively. When, however, the finger on the resistance A_3B_3 gets for the second time to the end of its scale we disconnect altogether the wires connected with P_3 , and move the wire connected with P_2 to P_3 and the wire connected with Q_1 to Q_2 . In working the device these operations seem quite natural and little thinking is required. It is also easy to see that if a one per cent. inaccuracy is permissible it is unnecessary to have more than three pairs of terminals on the bridge arms.

X. *Imaginary Roots.*

In solving certain engineering problems in connexion with finding the amplitudes, the damping factors, and the periods of certain mechanical and electrical oscillations, a necessary step is finding the imaginary roots of certain algebraic equations. Quadratic equations present no difficulty, and we have already shown how approximate values of the imaginary roots of cubic equations can be found.

With the biquadratic equations, however, which occur when discussing the theory of the parallel running of alternators*, the oscillations set up in coupled electric circuits in wireless telegraphy†, &c., both pairs of roots are sometimes imaginary. In this case we proceed as follows:—Let $x + yi$ be a root of the equation $f(z) = 0$. Then $f(x + yi) = 0$, and

* A. Russell, 'Alternating Currents,' vol. ii. p. 184.

† J. A. Fleming, 'Electric Wave Telegraphy,' p. 209.

hence, expanding by Taylor's theorem, we have

$$f(x) - \frac{y^2}{2} f''(x) + \dots + \frac{y}{2} \left\{ f'(x) - \frac{y^2}{3} f'''(x) + \dots \right\} = 0,$$

and thus we must have

$$\left. \begin{aligned} f(x) - \frac{y^2}{2} f''(x) + \frac{y^4}{24} f^{iv}(x) &= 0 \dots (a) \\ \text{and} \quad f'(x) - \frac{y^2}{6} f'''(x) &= 0 \dots (b) \end{aligned} \right\}.$$

From (b) we get y^2 in terms of x , and substituting this value of y^2 in (a) we get an equation of the sixth degree to find x . The two real roots of this equation can be found by the machine arranged in the manner shown in fig. 8, and the pairs of corresponding values of y are given at once by (b). Approximate values of the four imaginary roots can thus be rapidly found. If a higher degree of accuracy be required we can either use Newton's method of approximation, or better apply Horner's method to the auxiliary sextic.

Theoretically we can always obtain approximate values of all the roots real or imaginary of an equation of the n th degree by the device, as y^2 can always be easily eliminated from the equations corresponding to (a) and (b), by Sylvester's method*.

XI. *Solution of Equations containing Miscellaneous Functions.*

For this purpose some of the contact-fingers are made from wires bent into the shape of certain curves. Suppose, for example, we desire to find the roots of the equation

$$\frac{a}{x^m} + \frac{b}{f(x)} = cx^n + dF(x),$$

where the values of $f(x)$ and $F(x)$ have been computed, or found experimentally, for values of x .

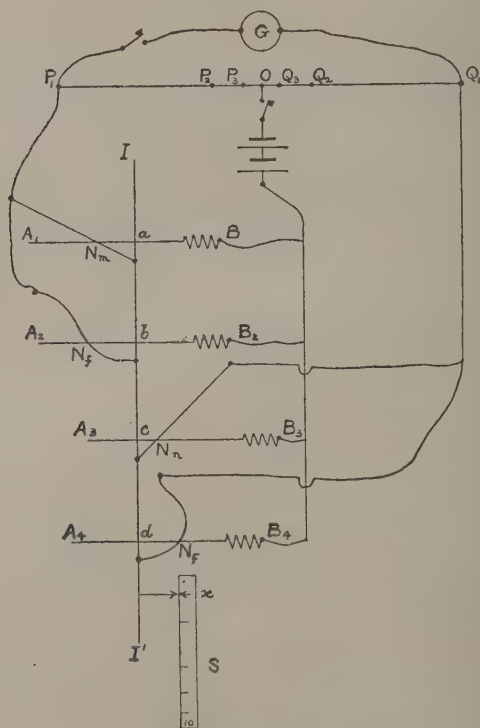
The wire passing through the point N_f (fig. 9, p. 818) is bent so that if y be a horizontal ordinate and x the vertical abscissa on the logarithmic scale, $y = h \log f(x)$, where h is the

* Burnside and Panton, 'Theory of Equations,' p. 296.

length of the scale on the slide resistances. Similarly the equation to the wire through N_F is $y = h \log F(x)$.

The contact-fingers through N_m and N_n are set so that their inclinations to the vertical are fixed at $-\tan^{-1} m$ and $\tan^{-1} n$ respectively. The rod II' is then gradually lowered and the readings on the logarithmic scale S , when the deflexion on

Fig. 9.



the galvanometer is zero, give the roots of the equation lying between 1 and 10.

The use of a bent contact-finger to take into account the "hysteresis loop" of iron would be of use in certain electrical problems.

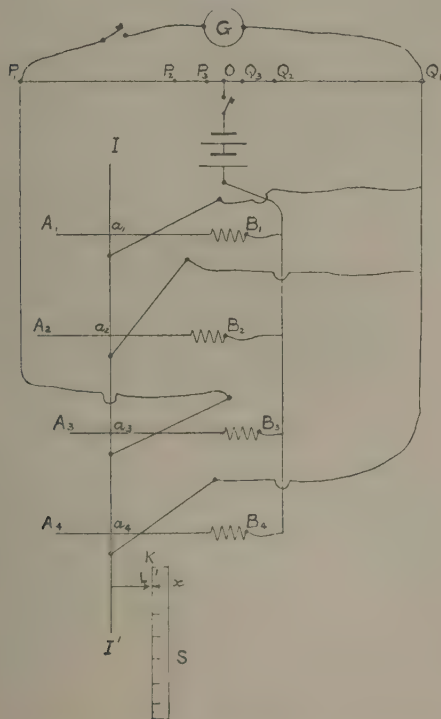
XII. *Solution of Transcendental Equations.*

Let us suppose that the equation we have to solve is

$$a_1 l_1^{b_1 x} + a_2 l_2^{b_2 x} - a_3 l_3^{b_3 x} + a_4 10^x = 0.$$

In this case (fig. 10) the scale S is an ordinary scale, the length KL being equal to hx . The various contact-fingers are adjusted so that the angles they make with the index-line II' are $\tan^{-1}(b_1 \log l_1)$, $\tan^{-1}(b_2 \log l_2)$, $\tan^{-1}(b_3 \log l_3)$

Fig. 10.



and 45° respectively. We connect the contact-fingers as in fig. 10 and vary x . The values of x for which there is no deflexion on the galvanometer are the roots of the given equation lying between 1 and 10. By altering the given

equation to one whose roots are ten times smaller we find the roots lying between 10 and 100, and proceeding in this way we can get approximate values of all the roots.

By similar settings of the contact-fingers, and by using both logarithmic and ordinary scales, approximate values of the roots of very complicated equations can sometimes be easily found.

XIII. *Tracing Curves Electrically.*

Suppose we desire to find the graph of the curve

$$y = a_p x^p + a_q x^q - a_r x^r + \dots = f(x),$$

where p, q, r, \dots may be positive, fractional, or negative indices. We set the contact-fingers at angles $\tan^{-1} p, \tan^{-1} q, \dots$ with the index-line, and move the slide resistances until the readings on the scales are a_p, a_q, \dots . The contact-fingers on the slide resistances representing the positive terms are then connected with P, and the other contact-fingers with Q. In addition we have a slide resistance Y with a vertical finger, which is also connected with Q. The fingers are moved down through a given distance x on the logarithmic scale, and the value of the reading y on the slide resistance Y when there is no deflexion of the galvanometer is then found. In this way simultaneous values of x and y can be rapidly obtained, and hence we can readily plot the curve.

The curve could also be traced automatically by making the spot of light from a mirror galvanometer, connected between P and Q, fall on a strip of sensitive paper which is constrained to move so that its velocity is always proportional to the rate at which x is increasing. The points where the trace cuts the line of zero deflexion would give the roots of the equation $f(x) = 0$, and the turning points of the trace would give the roots of $f'(x) = 0$. In the same way the integral curve

$$y = \int_0^x f(x) dx,$$

could be drawn automatically.

If the slide resistances were made of large size so that they could carry appreciable currents, recording ammeters and voltmeters could be employed to trace the curves.

XIV. *Conclusion.*

We think that if engineers and physicists recognize that approximate values of roots of very complicated equations can be easily obtained, it may considerably extend the usefulness of theory. Authors are often diffident to publish results which can only be utilized when the roots of equations of degrees higher than the third can be found, or when equations involving miscellaneous functions can be solved. In these cases we hope that a knowledge of the methods of using logarithmic slide resistances described above may be of practical value by showing how the theoretical results can be immediately utilized.

We hope also that this preliminary sketch will induce others to improve the method, and to apply it to other and possibly more practical uses. It seems, for example, particularly suited to harmonic analysis as the integrals representing the coefficients of $\sin nx$ and $\cos nx$ in the expansion of $f(x)$ can be readily found.

In conclusion, we shall quote from the quaint but spirited preface to Seth Partridge's book on the slide rule (1671).

"I am sure here is a good Subject, a good piece of Cloath, if the Garment be not marred in the making; if it be, the fault is in the botching Taylor, not in the stuffe."

DISCUSSION.

Prof. C. H. LEES expressed his interest in the device and referred to the large number of calculations that could be performed with it.

Dr. W. H. ECCLES congratulated the Authors, and referring to the fact that the machine could be used to solve a bi-quadratic, asked if it was possible to determine the two quadratic factors by means of it.

LVI. *The Echelon Spectroscope, its Secondary Action, and the Structure of the Green Mercury Line.* By HERBERT STANSFIELD, D.Sc., Demonstrator in Physics, Manchester University*.

Thesis approved for the Degree of Doctor of Science in the University of London †.

[Plates XXX.-XXXII.]

Part I.—THE ECHELON SPECTROSCOPE.

The Echelon and its Mounting.

THE echelon spectroscope described in this paper was constructed by Messrs. Adam Hilger for Professor Schuster, a modification, which has proved to be very valuable, being made in the usual design. A front elevation, plan, and two end elevations of the echelon are reproduced on Plate XXX., the plan being drawn with the cover removed.

There are 33 glass plates. The smallest plate is 13 mm. wide, and they increase in width by steps of 1 mm. until the last plate is reached, which is 12 mm. wider than the last but one, the aperture being reduced to 1 mm. by the screen S. The effective aperture of the first plate is similarly reduced in width to 1 mm. by the block B. All the plates are 40 mm. high, and the common thickness is 9.48 mm. The plates are pressed together by the two nickel steel rods marked T, which are intended to have the same coefficient of expansion as the glass. The plates are in very close contact, in most cases the greater part of an interface being taken up by a patch of definite but irregular outline that appears "black" by reflected light. The remaining part of the interface generally shows white of the first order, but here and there the film of air may be thick enough to show the yellow or even the red of the first order.

* Read June 11, 1909.

† The following alterations have been made:—Equations 2, 3, 4, 9, 9A, and the section on the spectrum given by a hot lamp have been added; also the faint lines previously described as doubtful are shown to have their origin in the echelon.

The common thickness of the glass plates is 9.48 mm.; their refractive index, deduced by a Hartmann formula from the values given by the makers of the glass, is 1.5802 for the green line 5461; and the dispersive power, $\frac{d\mu}{d\lambda}$, is -918 per cm. for this wave-length.

Optical Effects due to clamping the Plates.

The echelon produces a slight cylindrical convergence in a beam of light, altering the focus of the observing telescope by 1 mm. (the focal length of the object-glass being 53 cms.). This effect is probably due to the clamping, as Twyman* states that when the clamping pressure is applied a change of focus is produced. He attributes the effect to a uniform increase in the compression of the plates from the largest to the smallest; but there is direct evidence (see p. 830) that the echelon plates become slightly prismatic, and this effect, increasing towards the smaller end, would also produce convergence.

The focus is altered by rotating the echelon about a vertical axis. The convergence is increased, as would be expected, by turning the echelon so as to reduce the width of the emergent beam (see fig. 3 A), and diminished when it is turned so as to make the beam broader (fig. 3 B). Changes in the focus are also produced by covering some of the step-faces at either end of the echelon.

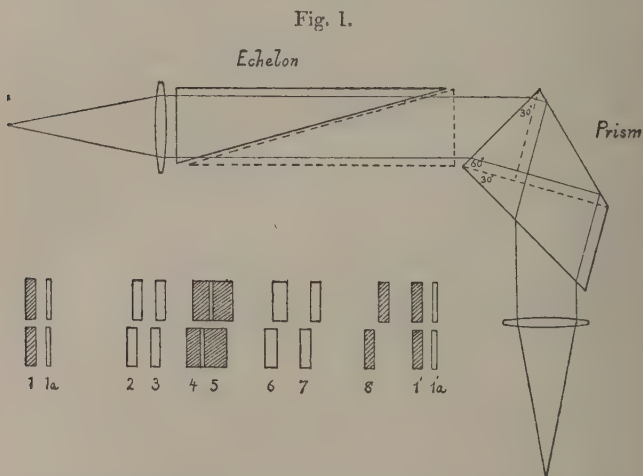
Use of Auxiliary Prism.

Auxiliary spectroscopes have generally been employed to pick out the particular line in the spectrum to be examined by the echelon, the slit of the echelon spectroscope being illuminated with the selected line; but a modification in the design of this instrument was made for Professor Schuster so that the prism from the auxiliary spectroscope could be mounted next to the echelon, as shown in fig. 1, the prism being made larger than usual in order to take the full width and height of the echelon beam. This arrangement, which appears to have been originally contemplated by Professor Michelson, has been found to have important advantages.

* Twyman, Proceedings of the Optical Convention, 1905, p. 53.

The dispersion produced by the prism, which is 2 per cent. of that given by the echelon, is subtracted from the echelon dispersion when the echelon is in the usual position shown by full lines, and is added to it when the echelon is in the reversed position shown by the dotted lines.

The change of 4 per cent. in the dispersion obtained in this way produces the alteration in the spectrum shown in fig. 1. The distance apart of successive orders of the same wave-length is not altered; but when the dispersion is reduced by the prism, all the lines belonging to the same order approach one another and draw away from the neighbouring orders. It is evident from fig. 1 that all but two of the lines



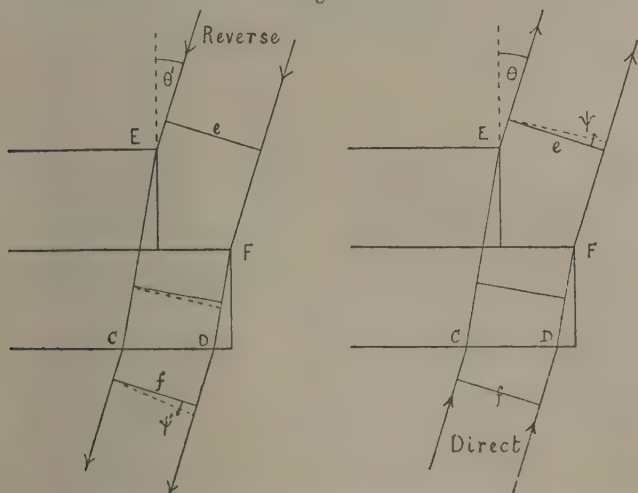
represented belong to the same order, and that these two lines, marked 1' and 1'a, belong to the next higher order. With the prism on an auxiliary spectroscope, no evidence of this kind is obtained; and it has simply been assumed in some cases that the companion lines belonged to the bright central line nearest to them.

PRIMARY ACTION OF THE ECHELON IN THE DIRECT AND REVERSED POSITIONS.

Equations for the Principal Maxima.

The theory of the echelon given by Michelson* for the case of light falling normally on the larger end, has been extended by Galitzin†, who has investigated, both by calculation and experiment, the motion and changes in the dispersion of the spectra when the echelon is rotated through small angles within the limits of two or three degrees on either side of the normal position. As the theory of the echelon in the reversed position has not, as far as I know, hitherto been considered, a comparison will be made below with the theory of the ordinary action.

Fig. 2.



In fig. 2 the reversed and direct cases are represented diagrammatically. The rays and wave-fronts drawn with full

* A. A. Michelson, *Astrophysical Journal*, viii. pp. 36-47 (1898). *American Journal* [4] v. pp. 215-217 (1898). *Journ. de Phys.* [3] viii. pp. 305-314 (1899).

† Furst B. Galitzin, *Bulletin de l'Académie Impériale des Sciences de St. Pétersbourg*, 5 série, t. xxiii. pp. 67-118 (1905).

lines are those regularly refracted, the dotted wave-fronts are drawn perpendicular to the directions in which a maximum of order n is formed. The angle between this direction and the normally refracted rays is called ψ in the direct case and ψ' in the reversed case. The distance apart in the air of regularly refracted rays passing through corresponding points E and F of neighbouring step-faces is called f at the end of the echelon and e on the step side. The condition for a principal maximum is that the sum of the distances of E from the incident wave-front on one side and the dotted wave-front on the other, shall be n wave-lengths greater than the sum of the distances of the corresponding point F from the same wave-fronts.

When the angles ψ and ψ' are sufficiently small, we may employ Fermat's principle and measure the optical paths along the regularly refracted rays. This gives the general equations in the form

$$R - e\psi = n\lambda, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R - f\psi' = n\lambda, \quad . \quad . \quad . \quad . \quad . \quad (1A)$$

where R is the retardation produced in a regularly refracted ray by its passage through a single plate. Here the equations referring to the reversed case are distinguished by a letter A.

The values of R , e , and f are given below, and are plotted in fig. 4 (p. 831).

$$R = t \left(\mu \sqrt{1 - \frac{\sin^2 \theta}{\mu^2}} - \cos \theta \right), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$e = s \cos \theta + t \sin \theta, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$f = s \cos \theta + \frac{t}{\mu} \sin \theta - \frac{\cos \theta}{\sqrt{1 - \frac{\sin^2 \theta}{\mu^2}}}, \quad . \quad . \quad . \quad (4)$$

Here s is the width of the step-faces, t the thickness of the plates, and θ the angle of incidence of the light on the plates. This angle is in practice generally kept within the narrow limits of $\pm 2^\circ$, and it is sufficient to retain only the lowest

powers of θ . These expressions then become

$$R = R_0 \left(1 + \frac{\theta^2}{2\mu} \right), \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$e = s + t\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$f = s + t \frac{\theta}{\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

R_0 (the value of R when $\theta=0$) is $(\mu-1)t$. The parabolic expression for $R - R_0$ in equation (5) gives a very close approximation to the value given by (2), the difference, which depends on θ^4 , being only 1 part in 800 when $\theta=5^\circ$. The slight deviations of e and f from the linear expressions (6) and (7) may be detected in fig. 4.

Substituting the approximate expressions for R , e , and f in equations (1) and (1A), we obtain the general equations in the form:

$$R_0 \left(1 + \frac{\theta^2}{2\mu} \right) - s\psi \left(1 + \frac{t}{s} \theta \right) = n\lambda, \quad . \quad . \quad . \quad (8)$$

$$R_0 \left(1 + \frac{\theta'^2}{2\mu} \right) - s\psi' \left(1 + \frac{t}{s} \frac{\theta'}{\mu} \right) = n\lambda, \quad . \quad . \quad . \quad (8A)$$

Equation (8) is in agreement with Galitzin's calculated values and has the support of his measurements which were made on orders close to the position of greatest brightness so that the angle ψ was always small.

If the principal maxima several places removed from the direction of the regularly refracted rays are considered, it becomes necessary to take into account the terms depending on ψ^2 , which were neglected in applying Fermat's principle. When the path of the diffracted light is measured along the diffracted rays, the equation for the direct case may be written

$$R - e \sin \psi + (t \cos \theta - s \sin \theta)(1 - \cos \psi) = n\lambda. \quad . \quad (9)$$

If we now suppose the light to retrace its path, the angle of incidence, θ' , on the step-faces is equal to $\theta - \psi$ (see fig. 2), and the angle $\theta' + \psi'$ at which the diffracted rays leave the end plate is equal to θ . Hence the equation for the reversed

case may be found by substituting $\theta' + \psi'$ for θ and ψ' for ψ in equation (9) after substituting for R its value from equation (2). The equation thus obtained may be written in the form:

$$t(\sqrt{\mu^2 - \sin^2(\theta' + \psi')} - \cos \theta') - s\{\cos \theta' \sin \psi' - \sin \theta'(1 - \cos \psi')\} = n\lambda. \quad (9A)$$

If the terms whose order in θ and ψ is higher than the second are neglected, equations (9) and (9 A) reduce to

$$R_0\left(1 + \frac{\theta^2}{2\mu}\right) - s\psi\left(1 + \frac{t}{s}\theta - \frac{1}{2}\frac{t}{s}\psi\right) = n\lambda. \quad (10)$$

$$R_0\left(1 + \frac{\theta'^2}{2\mu}\right) - s\psi'\left(1 + \frac{t}{s}\frac{\theta}{\mu} + \frac{1}{2}\frac{t}{s}\frac{\psi'}{\mu}\right) = n\lambda. \quad (10A)$$

These equations only differ from (8) and (8 A) by the addition of small terms depending on ψ^2 and ψ'^2 , whose values are given in Table I. below.

*Position of the Orders in the Field of View
and Dispersion.*

According to the first approximation formulæ (1) and (1A)

$$\psi = \frac{R - n\lambda}{e} \quad \text{and} \quad \psi' = \frac{R - n\lambda}{f}.$$

R , e , and f depend on θ but not on ψ ; so the orders are equally spaced and the dispersions, given by

$$\frac{d\psi}{d\lambda} = -\frac{n - t\frac{d\mu}{d\lambda}}{e} \quad \text{and} \quad \frac{d\psi'}{d\lambda} = -\frac{n - t\frac{d\mu}{d\lambda}}{f},$$

do not vary along the spectrum.

The second approximation formulæ (10) and (10 A) give

$$\psi = \frac{R - n\lambda}{s\left(1 + \frac{t}{s}\theta - \frac{1}{2}\frac{t}{s}\psi\right)} \quad \text{and} \quad \psi' = \frac{R - n\lambda}{s\left(1 + \frac{t}{s}\frac{\theta}{\mu} + \frac{1}{2}\frac{t}{s}\frac{\psi'}{\mu}\right)}$$

If the echelon is rotated a little, so that one order, say the m th, is in the position of maximum brightness where

ψ or $\psi' = 0$,

$$\psi = \frac{m-n}{1 + \frac{t}{s}\theta - \frac{1}{2}\frac{t}{s}\psi} \left(\frac{\lambda}{s}\right) \text{ and } \psi' = \frac{m-n}{1 + \frac{t}{s}\frac{\theta}{\mu} + \frac{1}{2}\frac{t}{s}\frac{\psi'}{\mu'}} \left(\frac{\lambda}{s}\right)$$

Table I. gives the values of the small terms in the denominators depending on ψ and ψ' for the first five orders on either side of the central order.

TABLE I.

$m \sim n$.	$\frac{1}{2} \frac{t}{s} \psi$.	$\frac{1}{2} \frac{t}{s} \frac{\psi'}{\mu}$.
1	·003	·002
2	·005	·003
3	·008	·005
4	·010	·006
5	·013	·008

The Wave-Length Intervals between the Orders.

The repetition of a line in a number of orders provides an echelon spectrum with a wave-length scale of nearly equal divisions, and the wave-length value of these divisions, $\Delta\lambda$, is a constant of the echelon which can be calculated for any wave-length from the thickness of the plates and the refractive indices given for the glass. The expressions for $\Delta\lambda$ for the direct and reversed cases may be obtained from the general equations. Neglecting the terms depending on θ^2 , which do not affect the values by more than one part in ten thousand, they both reduce to

$$\Delta\lambda = \frac{\lambda}{n-t} \frac{d\mu}{d\lambda}$$

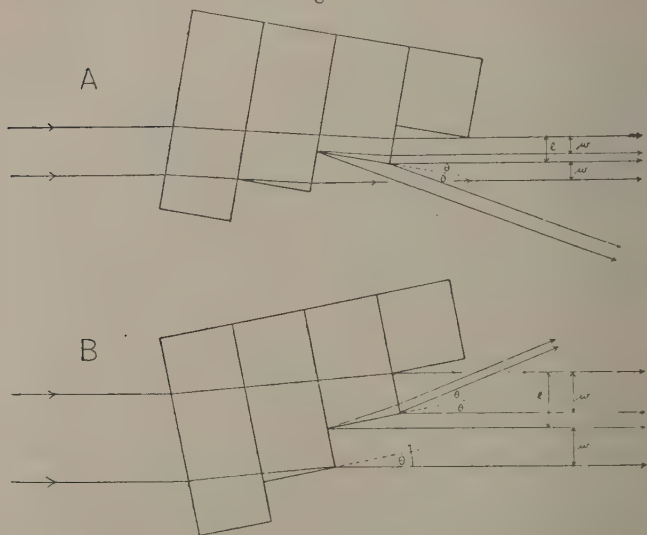
The changes that are made in n by rotating the echelon are so small compared with its whole value (10,070 for this echelon and the green mercury light) that $\Delta\lambda$ is sensibly constant for all positions. On the other hand, $\Delta\lambda$ may be increased or decreased by 2 per cent. by means of the auxiliary prism, as described on p. 824.

NOTES ON CHANGES PRODUCED BY ROTATING THE ECHELON
ABOUT A VERTICAL AXIS.

Changes in brightness, and the position of greatest brightness — As an order is moved across the field of view by rotating the echelon, it crosses the lateral maxima of the distribution of light due to the individual step apertures, disappearing as it crosses the diffraction minima, and having its greatest brightness as it crosses the central diffraction maximum. This position of greatest brightness corresponds to the direction of the regularly refracted rays, and is the origin from which the angle ψ is measured. It does not quite coincide with the position of the image of the slit when the echelon is removed, the position of greatest brightness being displaced towards the side on which the step-faces of the echelon lie. This displacement indicates that the echelon-plates are slightly prismatic.

Changes in dispersion and retardation.—Some of the effects produced by rotating the echelon about a vertical

Fig. 3.

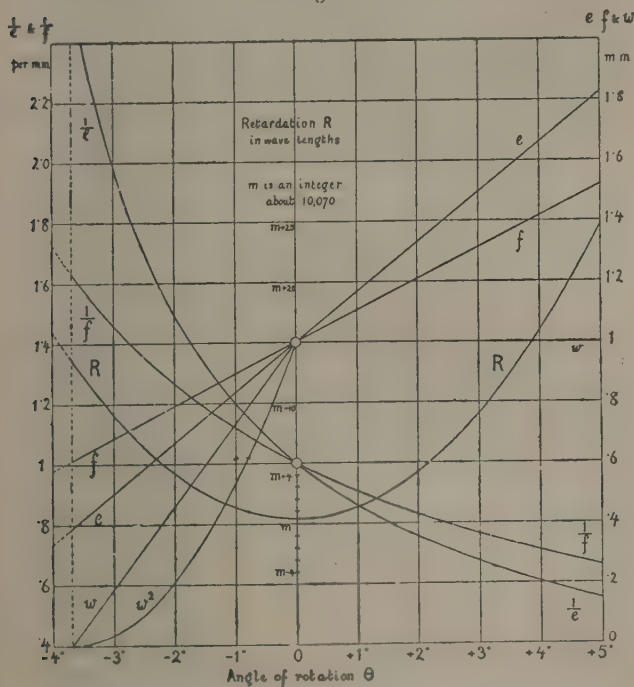


axis are represented in fig. 3. The echelon is shown at A

rotated so as to make θ negative; in this case e is smaller and therefore (see p. 828) the dispersion is larger than in the normal position. B shows the echelon rotated in the opposite direction so that e is increased and the dispersion decreased.

The way in which R , e , and f change with θ is represented in fig. 4. The scale for R is given in wave-lengths, m being the number of wave-lengths in the retardation of the order

Fig. 4.



which is nearest to the position of greatest brightness on the lower retardation side when $\theta = 0$. The positions of the row of marks on the $\theta = 0$ line represent, by their distances from the apex of the curve, the distances of the orders near the m th from the position of greatest brightness when $\theta = 0$, and the abscissæ of the curve passing through the marks which

lie within it, give the angle through which the echelon must be turned in either direction in order to bring orders higher than the m th up to the position of greatest brightness.

The reciprocals of e and f are also plotted to show how the dispersion is changed by the rotation in the direct and reversed cases. These changes in dispersion are simply changes in magnification, as they are not accompanied by any sensible change in the wave-length intervals between the orders.

The way in which the effective width, w , of the individual step apertures is reduced when the echelon is rotated so as to make θ negative, is represented in figs. 3 and 4; w^2 is also plotted in fig. 4, to indicate the falling off in the intensity of the light on this account.

Width of the diffraction-bands.—It will be seen from fig. 4 that both e and f are greater than w , except when $\theta=0$.

Hence, the angular interval, $\frac{\lambda}{w}$, between the equally spaced diffraction minima is in general greater than $\frac{\lambda}{e}$ or $\frac{\lambda}{f}$, the angular intervals, in the direct and reversed cases, between successive orders.

Light reflected from the ends of the plates.—The step ends of the echelon plates, which are finely ground, scatter some of the light falling upon them and also give regularly reflected beams, represented in fig. 3. The reflected beams produce echelon spectra; but as the ends are not polished, the spectra are poorly defined and the spreading of the light brings into greater prominence the broad diffraction-bands corresponding to the narrow sources represented by the illuminated end-faces of the plates. When the echelon and prism are used together, as in fig. 1, the reflected spectrum of the green mercury-line may be made to overlap the direct spectrum of the violet line by giving θ a small negative value. I think it would be an advantage to have the ends of the plates polished or blackened.

The position of minimum deviation.—As the echelon is rotated from one side of the normal position to the other, the orders first move across the field of view in the direction of decreasing deviation (measured by ψ) and then turn round and go back again. If the value of R happens to be a whole

number of wave-lengths for normal incidence, say $m\lambda$, then the m th order will come to its position of minimum deviation in the position of greatest brightness when $\theta=0$, but higher and lower orders will not then be quite in their positions of minimum deviation. Consider, for example, a lower order: ψ (or in the reversed case ψ') will be positive, and its value when $\theta=0$ can be reduced a little by increasing θ (in the positive direction), as at first the value of R is almost stationary, while e is increasing, and therefore the dispersion is decreasing.

By differentiating the general equations (1) and (1A), it will be found that the conditions for the turning-points in the direct and reversed cases are,

$$\frac{dR}{d\theta} = \psi \frac{de}{d\theta} \quad \text{and} \quad \frac{dR}{d\theta} = \psi' \frac{df}{d\theta}.$$

Substituting the values of the differential coefficients and calling the minimum values of the deviation in the two cases ψ_M and ψ'_M and the corresponding angles of incidence θ_M and θ'_M ,

$$\theta_M = \frac{\mu}{\mu-1} \psi_M \quad \text{and} \quad \theta'_M = \frac{1}{\mu-1} \psi'_M.$$

Hence as the echelon is rotated, so as to increase θ , the orders higher than the m th come to their minimum deviation positions before the echelon is normal to the light, and the lower orders have their minimum deviation after the normal position is passed. The central orders are very nearly in their positions of minimum deviation when $\theta=0$. If ψ_0 and ψ'_0 are written for the values of ψ and ψ' in this case, it may be shown that

$$\frac{\psi_0 - \psi_M}{\psi_M} = \frac{1}{2} \frac{t}{s} \frac{\mu}{\mu-1} \psi_M \quad \text{and} \quad \frac{\psi'_0 - \psi'_M}{\psi'_M} = \frac{1}{2} \frac{t}{s} \frac{1}{\mu-1} \psi'_M,$$

and the values calculated from these formulæ show that, when this echelon is in the direct position, $\psi_0 - \psi_M$ is .7 per cent. of ψ_M for an order one place from the position of greatest brightness, 1.4 per cent. of ψ_M for an order two places away, and so on. The corresponding values for the reversed position are .4 per cent. and .9 per cent.

Effects produced by Temperature Changes.

The position of the various orders in the field of view when the echelon is in a definite position, such as the normal position, depends on the temperature of the echelon and on the refractive index of the surrounding air. Records of the temperature of the echelon and micrometer readings of the position of the orders in the field of view, taken from day to day, show that a rise of temperature of 8.6° C. moves the spectrum through a distance equal to the interval between neighbouring orders, indicating an increase of one wavelength in the retardation, R , for this rise of temperature.

Curving of the Spectrum Lines.

The echelon spectrum lines, like those of a prism spectrum are curved with the concave side toward the violet end. The curving is due to the variation in the angle of incidence, and therefore in the retardation, R , of light from different points along the length of the slit. The theory has been given by Laue*.

Consider the simplest case, in which the slit is vertical and the plates are normal to the axis of the collimator: then light from points above and below the centre of the slit will have a vertical plane of incidence on the plates. Writing i instead of θ for the angle of incidence in equation (3),

$$R - R_0 = \frac{1}{2} t \frac{\mu - 1}{\mu} i^2;$$

but from equation (1),

$$R - R_0 = e(\psi - \psi_0);$$

and in this case $e = s$, so that

$$\psi - \psi_0 = \frac{1}{2} \frac{t}{s} \left(\frac{\mu - 1}{\mu} \right) i^2. \quad . \quad . \quad . \quad (8)$$

Hence, for small values of i , the spectrum lines are parabolas. The curvature is very small. If, for example, the extreme values of i are ten times the angular separation of the orders,

* *Physikalische Zeitschrift*, vi. pp. 283-285 (1905).

$\Delta\psi$, then the top and bottom of the curved image of the slit, representing one order of the spectrum, will be 2 per cent. of $\Delta\psi$ to the violet side of the centre of the image.

Effects produced by Rotating the Echelon about a Horizontal Axis parallel to the Plates.

The echelon can readily be tilted in this way by placing a block under the foot at the small end. The block which I employ tilts the echelon through an angle of nearly 3° , and the photographs Nos. 1 to 3 in Pl. XXXI. of the green mercury line were taken with this angle of tilt. The horizontal axis of the parabolic lines, which passes through the normal to the plates, has been raised much above the field of view of these photographs, so that the lines where they cross the field of view are considerably inclined to the vertical. The reproductions in Pl. XXXI. have the centre of the curves below them, as they have been turned round in order to put the shorter wave-lengths on the left.

Part II.—SECONDARY ACTION OF THE ECHELON.

Secondary Bands in the Primary Spectrum.

The inclined spectrum lines of photograph No. 1, Pl. XXXI. are broken up and have a ropy or screw-like appearance because they are crossed by a secondary system of bands. This screw-like structure of echelon spectrum lines was observed by Gehrecke*. He does not explain how the new bands are produced, but shows that the appearance can be imitated by tilting an echelon with only two apertures, so that the echelon bands slope across the central vertical diffraction band.

When the echelon is in the ordinary position the secondary bands are parallel to the spectrum lines, and so their effects, though very important, are not so easily recognized as they are in this photograph taken with the echelon tilted.

Character of the Secondary Bands.

The secondary bands are superposed on the echelon lines and resemble them in appearance; they are also affected in the

* *Annalen der Physik*, xviii. p. 1074 (1905).

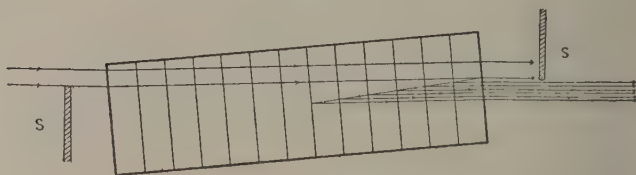
same way, but to a greater extent, by adjustments of the echelon. When the echelon is rotated, for example, the secondary bands move faster than the spectrum lines in the same direction and so move across them. When the echelon plates are vertical, the secondary bands, like the echelon lines, are vertical in the centre of the field; they are also curved in the same direction but more strongly. Their width is about the same as that of the finer spectrum lines, and so, in the ordinary position of the echelon, they are not easily recognized; but when the echelon is tilted they become more inclined than the spectrum lines in the same direction and can then be seen as in photograph No. 1, Plate XXXI., intersecting all the spectrum lines.

The behaviour of the secondary bands suggested the idea that they might be spectrum lines of a higher order, such as might be produced by the reflexion of light in the echelon.

Fabry and Perot Spectrum produced by the Secondary Action.

The secondary light, which has been twice reflected in the echelon, is by no means negligible. The echelon was tilted as shown in fig. 5, screens SS being arranged to cut out the

Fig. 5.



primary light, and it was found that the echelon was bright with secondary light coming out below the second screen, the brightness extending down to the bottom of some of the step apertures.

Suppose that each interface reflects a very small proportion of the light falling upon it and leave out of account for a moment the step structure of the echelon. There will be a secondary beam produced by the combination of the n faint secondary beams which have gone back through one plate, n

being the number of plates coming into action; let the retardation of this beam be Δ . There is another secondary beam whose retardation is 2Δ , made up of $n-1$ faint beams which have each gone back through two plates, and so on. Hence the secondary action of the pile of plates in the echelon is similar to that of a Fabry and Perot spectroscope, and, under suitable conditions of illumination, the secondary light by this action would be thrown into a ring spectrum of a high order. The retardation is $2\mu t$ for normal incidence, so it is $\frac{2\mu}{\mu-1}$ times as great, in this case about five and a half times as great, as that of the primary spectra.

The secondary light also undergoes the ordinary echelon treatment as it leaves by the step-faces, and it is therefore confined to the primary spectrum lines.

The photograph No. 2, Pl. XXXI., which shows short portions of the rings of the Fabry and Perot spectrum crossing the echelon spectrum, was obtained by stopping the primary light in the way described above, and making the echelon lines broad by widening the slit. The spectrum lines produced by the primary action of the echelon upon the secondary light are inclined because the echelon is tilted as shown in fig. 5, and the centre of the ring system is above the field of view for the same reason. The echelon has also been rotated (in the positive direction of θ), so the centre of the ring system is displaced laterally as well as vertically.

The Fabry and Perot spectrum lines are not dependent for their definition, like the echelon spectrum lines, on the narrowness of the slit; their want of clearness in this photograph is partly due to the overlapping of lines belonging to different orders, which is produced by the orders overlapping four deep.

The Secondary Point Spectrum.

If the slit, instead of being opened wide to show portions of the rings, is made narrow enough to give good definition to the primary echelon action, the secondary light which has undergone both actions is confined to dots indicating the position of the points of intersection of the spectrum lines in

one system with corresponding lines, that is lines representing the same wave-length, in the other system.

The horizontal dispersion of the dots, given by the ordinary echelon action, now prevents overlapping, as the wave-length interval between the orders in this system is a little greater than the length of the spectrum, and the secondary light gives in this point spectrum, therefore, the advantages of a very high order without the usual overlapping.

Gehrcke and Baeyer * obtained similar spectra which they call interference points, by crossing plane parallel plates, and have pointed out the advantages of combining two independent high dispersions.

A photograph of the secondary point spectrum, obtained with an exposure of one hour, is reproduced in Plate XXXI., No. 3. The echelon had the usual tilt, about 3° , and the echelon table was rotated about 2° in the positive direction of θ from the normal position. The dispersion in the Fabry and Perot system is, in this case, twelve times that in the echelon system, so the wave-length intervals can be best determined by the position of the dots in the former system, while the dots can be recognized and their wave-lengths can be roughly fixed by their position in the latter system.

There was no difficulty in recognizing the dots marked 1, 2, 3, 4, 5, 6, and 7 in the photograph, which represent well-known lines in the green line spectrum.

In order to determine the wave-length intervals, it is necessary to calculate $\Delta\lambda$, the wave-length interval between neighbouring orders of the Fabry and Perot spectrum. It was found from the formula

$$\Delta\lambda = \frac{\lambda}{p - 2t \cos r \frac{d\mu}{d\lambda}},$$

where p , the order of the spectra, is given by

$$p = \frac{2\mu t \cos r}{\lambda},$$

r being the angle of refraction of the light in the plates, and t their thickness. The value found in this way for the centre

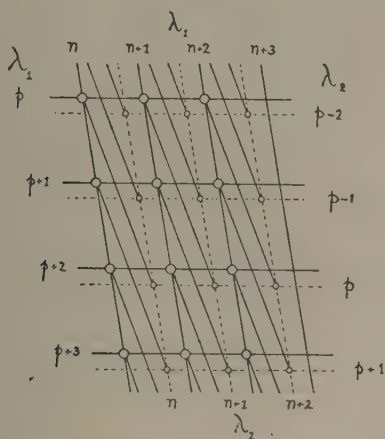
* E. Gehrcke and O. v. Baeyer, *Annalen der Physik*, xx. p. 269 (1906).

of the photograph is $96.5 \text{ m.}\text{\AA}$. The wave-length intervals between the components deduced from the measurements of the photograph and this value of $\Delta\lambda$ are given in Table III., p. 393; most of them agree closely with the values which have been found by other methods.

*Character of the Point Spectra produced by the
Secondary Light.*

Fig. 6 is a diagram representing the type of spectrum produced by the secondary light when the echelon is in the direct position and tilted about a horizontal axis, that is, the

Fig. 6.



type represented in the photograph of the secondary point spectrum No. 3, Pl. XXXI.; but in order to make the diagram clearer, the ratio of the dispersion in the Fabry and Perot system to the dispersion in the primary echelon system has been made much smaller than it is in the photograph.

In this case all the rays of light from a given point of the slit are parallel to one another during their passage through the plates, as the lateral diffraction does not take place until the light leaves the echelon by the step-faces. The spectra, or lines of constant retardation, in the Fabry and Perot system are therefore drawn in the diagram as horizontal lines.

When the photograph was taken the echelon had been rotated (from the normal position about a vertical axis) as well as tilted, so the plane parallel to the axis of the collimator and the diffracting apertures would not be quite vertical, but the deviation from the vertical is a small angle of the second order (equal to the product of the small angles of tilt and rotation), and it will be seen that lines joining two dots representing the same wave-length and of the same order in the Fabry and Perot system, would be sensibly horizontal, although if the slit had been opened wide so as to show portions of the Fabry and Perot rings, they would have been inclined about 45° , as in photograph No. 2.

When the echelon is reversed, the lateral diffraction takes place as the light enters the echelon and the lines of equal retardation in the Fabry and Perot system will be represented by circles whose centre is the point in the image plane corresponding to the direction of the normal to the plates.

The four long inclined lines in the diagram (fig. 6) represent spectra in the primary echelon system; they are inclined to the vertical because the echelon is tilted. The orders of a wave-length λ_1 are represented in the two systems by full lines, the dotted lines representing similarly the spectra of a second wave-length λ_2 greater than λ_1 . The order of the spectra in the two systems is indicated at the ends of the lines. The points where the full lines of the two systems intersect give a series of points (marked by the larger circles in the diagram) which represent λ_1 in the joint system, each point being defined by two orders. The top left-hand point in fig. 6 may be described for example as the np order of wave-length λ_1 . In the same way the intersections of the dotted lines give the positions where λ_2 is represented in the joint system, and the shorter inclined lines joining the λ_1 and λ_2 points of the same denomination represent the appearance of the joint spectra corresponding to a spectrum continuous between these limits.

It will be seen that there is no chance of spectra of different denominations overlapping in the joint system as long as there is no overlapping in one of the two systems which are combined.

The spectra in the two systems may be regarded as forming

an oblique system of coordinates ; the echelon system giving horizontal dispersion may be called the X system and the Fabry and Perot system the Y system. Then the slope of the spectra in the joint system, $\frac{dy}{dx}$, is the ratio of $\frac{\partial \lambda}{\partial x}$, the dispersion in the X system, keeping y constant, to $\frac{\partial \lambda}{\partial y}$, the dispersion in the Y system, keeping x constant.

One important feature of these point spectra is that they give a system of lines whose definition depends in general on the defining power of the two systems which are combined, but does not depend on the smallness of the range of wave-length in the light examined, so long as that range does not exceed a certain relatively large limit.

If the definition is poor in one of the two systems, a monochromatic radiation would be represented by dots elongated in the direction of the spectrum lines of the other system, and this would in general spoil the sharpness of the lines representing in the joint system a spectrum continuous between narrow limits ; but if these lines are nearly parallel to the spectrum lines of one system, the want of definition in the other will not spoil the definition of the lines, as each dot representing a single wave-length will be drawn out in a direction nearly parallel to the length of the joint spectra. This special case is realized when the echelon is in the direct or reversed normal positions.

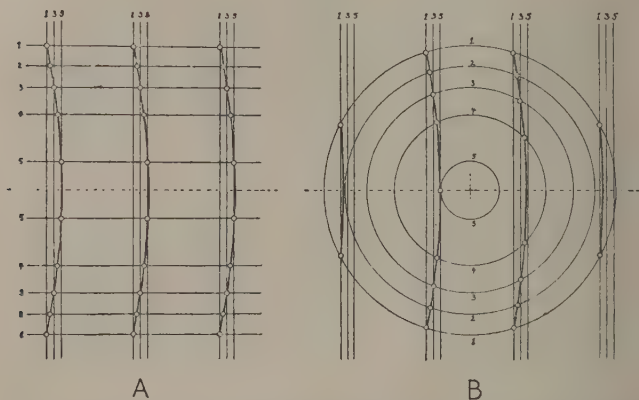
The diagrams A and B in fig. 7 represent the conditions in these cases. They are similar to fig. 6, but spectrum lines have been drawn in the Fabry and Perot system for a series of five wave-lengths whose values increase by equal increment from λ_1 to λ_5 . In the primary echelon system the lines representing alternate wave-lengths have been omitted. B represents the case in which the echelon is in the reversed position, the radii of the circles, representing the spectra in the Fabry and Perot system, being chosen so that their squares increase by equal increments. The horizontal lines in diagram A represent the same spectra when the echelon is turned round into the direct normal position.

For convenience in drawing the diagrams, the dispersion in the Fabry and Perot system has been represented as only

about ten times as great as that in the primary echelon system. It should be several hundred times as great even for those parts of the diagram farthest from the centre.

It will be seen that in both diagrams the secondary point spectra, drawn through the points of intersection of lines

Fig. 7.



representing the same wave-length in the two systems which are combined, are curved so as to be concave towards the side of shorter wave-length in the primary echelon system; the curvature has been much exaggerated by taking the ratio of the dispersions so much smaller than it actually is.

Origin of the Secondary Bands.

There is no doubt that the secondary bands which are observed when the echelon is employed in the usual way, that is, with the secondary light superposed on the primary, are very closely connected with the point spectra produced by the secondary light. All the characteristics of the secondary bands described on page 836 may be explained by supposing that they are, like the point spectra of the secondary light, the loci of the points of intersection of lines representing the same wave-length in the primary echelon and Fabry and Perot spectra.

Another characteristic which may be explained in the

same way, with the help of fig. 7, is that when the echelon is in the direct normal position the secondary bands affect each order of the spectrum in the same way, while, when the echelon is reversed, there are marked differences between the different orders.

The secondary bands have other characteristics which may find their explanation in interference taking place between the secondary light and the primary. The dark secondary bands appear to cut through the bright primary spectra. This is shown to some extent in photograph No. 1, Pl. XXXI., but it is much more marked when the secondary light is made relatively stronger by covering half the echelon apertures so as to stop all the light which has passed through less than half the plates in the echelon. The brightness of the primary lines is no doubt increased above and below the dark bands, making them appear darker by contrast, but I think there is little doubt that there has been an actual reduction in the brightness of the primary light in the dark secondary bands.

Another feature of the secondary bands, which may be seen in photograph No. 1, Pl. XXXI., where they cross the line 1, is that they always appear in pairs, fainter and stronger bands alternating with one another. This may be connected with the difference in phase between the primary and the secondary light produced by the two reflexions of the latter. Apart from these phase changes at reflexion, the primary and secondary light would be in phase at the points of intersection of their maxima, as the retardations in both systems are whole numbers of wave-lengths in the direction in which the maxima are formed. There are two cases to be considered for the secondary light, according as it has undergone both reflexions at interfaces or one reflexion at an interface and the other at an external surface. In the former case, as the air-films are very thin, it is possible that a change of phase of $\frac{\pi}{2}$ may be introduced at each reflexion which would give the best phase conditions for interference between the maxima. The latter case may account for the second set of bands shifted relatively to the first.

PART III.—STRUCTURE OF THE GREEN MERCURY LINE.
(5461.)

*Description of Spectrum given by an Arons Lamp, and
Comparison with the Results obtained by other Observers.*

This spectrum consists of a bright principal line, which is a close double, with six companion lines, three on either side. A photograph of the primary echelon spectrum is reproduced in Plate XXXII. together with a diagram of the spectrum. The photograph shows, in addition to the genuine components, a number of faint lines which have their origin in the echelon. The genuine components are numbered from 1 to 8, and the false lines are marked 1 *a*, 1 *b*, 3 *a*, &c. The lines 1 *a*, 1 *b*, 1 *c*, &c., mark the positions of the secondary diffraction maxima on the longer wave-length side of the principal diffraction maximum 1, and the lines 3 *a*, 5 *a*, 6 *a*, 7 *a*, and 8 *a* represent the first secondary maxima on the longer wave-length side of the lines 3, 5, 6, 7, and 8. When the aperture of the echelon is reduced by covering the first ten step-faces at the smaller end, the faint lines move away from their parents into the new positions of the secondary maxima corresponding to the reduced number of apertures.

The numbers in Table II. and Table III. give the distances of the various components from the component of shortest wave-length. I adopted this method of measuring the positions because the bright central line appeared to be the most variable component, while the component of shortest wave-length is a good reference line. The results of the other observers, given in the Tables, are not convenient for comparison in their original form, because Gehrecke and Baeyer*, Janicki†, and Galitzin and Wilip‡, give the distances from the centre of the principal line, while Fabry and Perot§, Baeyer|| (in a later paper), and Nagaoka¶, divide

* F. Gehrecke and O. Von Baeyer, *Annalen der Physik*, vol. xx. p. 269 (1906). † L. Janicki, *Annalen der Physik*, vol. xix. p. 36 (1906).

‡ Fürst B. Galitzin und J. Wilip, *Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg*, sér. 8, vol. xxii. no. 1 (1906).

§ Astrophysical Journal, xv. p. 218 (1902).

|| O. v. Baeyer, *Verhandlungen der Deutschen Physikalischen Gesellschaft*, ix. no. 4, p. 84 (1907). ¶ Nagaoka, 'Nature,' vol. lxxvii. p. 582 (1908).

the principal line into two components and measure the distances from the brighter one.

Another difficulty in the comparison is that the values obtained by Gehreke and Baeyer, and later by Baeyer, by the use of crossed plane-parallel plates, are systematically higher than the values obtained with echelon spectroscopes, which agree fairly well amongst themselves. I have reduced all Baeyer's intervals 5 per cent., and the earlier values of Gehreke and Baeyer (the means of three sets of measurements given in their paper) by 3 per cent.; and it will be seen that, apart from these differences in the constants, the two independent methods are in close agreement.

TABLE II.—Measurements of the Green Mercury-line Spectrum.

The distances of the component lines from the component of shortest wave-length are given in milli-Ångström units.

Fabry & Perot.	Crossed Plates.		Echelon Spectroscopes.		Reference Numbers.
	Gehreke & Baeyer.	Baeyer.	Janicki.	Galitzin & Wilip.	
0.1 Å. units	reduced 3 p.c.	reduced 5 p.c.			
0	0	0	0	0	1
15	126	136	133	137	2
17	169	169	166	168	3
...	...	189	...	189	...
<u>22</u>	} <u>234</u> {	214	} <u>232</u> {	<u>236</u>	4
<u>23</u>		<u>238</u>			5
					} Central Band.
31	318	320	320	321	6
36	364	363	365	365	7
...	...	449	8

The brightest component in each case is underlined.

Fabry and Perot's values* given in the Table are those published by Zeeman† in 1902. They help to confirm

* M. Perot informs me (in a letter dated Oct. 13th, 1908) that they have not published any particulars of this line since then.

† Astrophysical Journal, xv. p. 218 (1902).

TABLE III.—Measurements of the Green Mercury-line Spectrum (*continued*).

Nagaoka.	Author.			Reference Numbers.
Echelon Spectroscop.	Primary Spectrum.	Secondary Point Spectrum.	Width.	
0	0	0	17	1
31
72
105
137	135	141	13	2
163	164	167	13	3
189
223	<u>232</u> { (217) (243)	217	16	4
<u>247</u>		<u>243</u>	24	5
280
315	319	322	17	6
356	363	365	12	7
390
448	448	...	14	8
477

the accuracy of the constants employed in the echelon calculations.

The results obtained by several other observers are given by Janicki*.

The series of faint lines given by Nagaoka resembles the series of false lines in my primary echelon photographs. I published my measurements for comparison with his before discovering that the faint lines in my photographs were not genuine †.

Width of the Components.

The mean values of the widths of the photographic images of the components are given in Table III. If the echelon acted perfectly, the width of a principal light maximum

* *Loc. cit.* p. 61.

† 'Nature,' vol. lxxviii. p. 8 (1908).

constituting one order of the primary echelon spectrum of a monochromatic radiation, with a narrow slit, would be $2/33$ of the interval between the orders, which corresponds in this case to a width of 30 m.Å. The narrowest lines in the photographs which still have a measurable width, are about 10 m.Å. wide. The widths of the brighter lines vary considerably with their exposure, while the width of the companion line 8, which is too faint to be overexposed, is the same on each of the three plates on which its width could be measured.

Measurement of Secondary Spectrum.

The results obtained by measuring the secondary point spectrum are also entered in Table III. The photograph measured was exposed for an hour, and is the one reproduced in Plate XXXI., No. 3. It will be seen that the agreement of these results, obtained by an independent method, with the ordinary echelon values, is fairly close, except for the component 2. The methods agree very closely as to the position of the central line. They both give the centre of the double line at 232 and the dividing dark line at 228. The two components of the central line were not measured separately in the primary spectrum photographs, but the position of the dark dividing line was sometimes recorded. The positions of the centres of the components given in brackets in the second column of Table III., are deduced from the position and width of the whole line and the position of the dark dividing line (neglecting its width).

Spectrum given by a Bastian Lamp.

A striking variation in the spectrum of the green line was observed with a Bastian mercury arc-lamp. The glass tube through which the discharge passes is bent nearly into the form of an S in a horizontal plane, so that when one part of the discharge is parallel to the slit plate, another part may be normal to it. When the image of the "end-on" portion of the discharge is put on the slit, the change in the spectrum is so great that it is difficult at first to recognize the components. On measuring a photograph of the "end-on" spectrum, however, it was found that the companion

lines keep their relative positions, although some become broader and brighter, but the dark space between 4 and 5, the two components of the principal line, is greatly increased (from 6 to 26 m.Å.) and, being no longer brighter or broader than the rest, they look like ordinary companion lines.

The "side-on" spectrum of the Bastian lamp resembles more nearly that given by the Arons lamp.

The separation of the components of the principal line in the "end-on" radiation has been investigated by Galitzin and Wilip*, who observed the phenomenon first in the case of a Geissler tube arranged so that the axis of the discharge was normal to the slit plate.

*Spectrum of the Green Mercury line given by a hot
Mercury lamp.*

Janicki† describes the broadening of the components of the green and yellow mercury lines which takes place when a mercury lamp is allowed to become sufficiently hot, and a peculiar system of five equidistant bands which he observed when the original components of the lines were lost in a continuous spectrum. Galitzin and Wilip‡, who give measurements of the bands, suggest that they may be due to a reversal of the lines, or perhaps to some peculiar property of the echelon spectroscope.

The theory of the secondary echelon spectra (see page 841) indicates that the secondary bands would be well defined in a short continuous spectrum, and it seemed probable that they were the bands which Janicki and Galitzin & Wilip observed.

I have tested this point with a quartz-lamp fitted with an air-manometer, similar to that described by Galitzin and Wilip§.

When an arc is started with the lamp cold, the central line (4 and 5 together) and all the companion lines are at

* Fürst B. Galitzin und J. Wilip, "Ueber die Eigenschaften einiger Emissionslinien des Quecksilberdampfes," *Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg*, sér. 8, vol. xxii. no. 1 (1907).

† *Loc. cit.* pp. 49-55.

‡ *Loc. cit.* pp. 34 & 76.

§ *Loc. cit.* p. 4.

first plainly visible, but the pressure in the lamp increases rapidly and the broadening and coalescing of the lines soon takes place. The position occupied by the bright central line in the various orders is now marked by a dark line, probably due to absorption. As the companion lines become merged in the general brightness the fine secondary bands become clearly visible in all the bright parts of the field, the fact that they are secondary bands being shown by their motion relative to the primary spectrum when the echelon is given a slight rotation. The secondary bands show clearly on the green line when the pressure in the lamp is about one atmosphere.

This research was commenced at the request of Professor Schuster, and I have much pleasure in acknowledging the help I have received from the interest he has taken in its progress.

I wish to thank Mr. E. Marsden for assistance in my first experiments on the secondary effects, and Mr. W. A. Harwood, B.Sc., for measuring several of the photographs. My thanks are also due both to Professor Schuster and to Professor Rutherford, for placing the resources of the Physical Laboratories at my disposal.

DISCUSSION.

Dr. LEES referred to the importance of the secondary action and asked the Author if it was now possible to say definitely whether a line observed in an echelon spectrum is genuine or is produced by the instrument.

The AUTHOR said that Gehrecke and Baeyer had hoped to supply the means of settling doubts of this kind when they eliminated the ghosts from their green line spectrum by their method of "interference-points." Since then, however, two faint lines had been added to the list of components. With the possible exception of one faint line agreement had now been arrived at between two independent methods.

LVII. *Inductance and Resistance in Telephone and other Circuits.* By J. W. NICHOLSON, M.A., D.Sc.; Trinity College, Cambridge*.

I. EFFECTIVE INDUCTANCE.

A GENERAL formula for the effective inductance of a circuit consisting of two long parallel wires has been given by the author †, and is suitable for cases in which the current distribution in either wire is greatly affected by the frequency of alternation. In its general form, although its limitations are clearly defined, the result is not well adapted, in the absence of tables, to rapid calculation. The main object of the present paper is to examine certain important cases in detail, and to obtain formulæ capable of immediate use. A calculation of the effective resistance is also made in each case. A problem to which attention has been mainly directed, which includes several practical cases of great interest, is that of the simple telephone circuit, in which the leads are not twisted round each other in order to annul the inductive effects of the earth and of neighbouring circuits.

In the proof of the general formula, the influence of electrostatic capacity was ignored. This imposes a great limitation upon the types of circuit for which the expression may be used. An estimation of the maximum capacity causing no alteration to a given order of accuracy is given in this paper.

Throughout the investigation, only iron and copper wires, as the two extreme cases, are considered. The large permeability of iron completely changes the character of the effect of frequency on its self-induction, as compared with other metals. To all metals except iron greatly used in practice, the formulæ developed for copper wires may be applied with a nearly identical order of accuracy.

Let a be the radius of either wire, c their distance apart, and (μ, σ) their permeability and resistivity. They are equal in all respects.

* Read June 11, 1909.

† Phil. Mag. Feb. 1909; Proc. Phys. Soc. vol. xxi.

The value of the inductance per unit length is then

$$L = 4 \log \frac{c}{a} + \frac{4\mu}{x} \cdot \frac{\text{ber } x \text{ber}' x + \text{bei } x \text{bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} + M, \quad (1)$$

where, if $n/2\pi$ be the frequency,

$$x = 2a(\pi\mu n/\sigma)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, M and N being real,

$$c^2(M + iN) = \frac{4a^2(\log a/c - \mu J_0/ka J_0')(1 - \mu J_1/ka J_1')}{(\log ka - \mu J_0/ka J_0')(1 + \mu J_1/ka J_1')} \quad (3)$$

The Bessel functions have an argument $ka = x\epsilon^{\frac{1}{2}}$, and if V be the velocity of propagation of electromagnetic disturbances in the outer medium,

$$h = n^2 a^2 c / V^2. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

M may be described as the correction for closeness of the wires.

The effective resistance per unit length is given by

$$R = \frac{4\pi n}{x} \cdot \frac{\text{ber } x \text{bei}' x - \text{bei } x \text{ber}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} - Nn. \quad (5)$$

Now the functions $\text{ber } x$, $\text{bei } x$ are usually defined by

$$J_0(x\epsilon^{\frac{1}{2}}) = \text{ber } x + i \text{bei } x,$$

so that

$$J_1(x\epsilon^{\frac{1}{2}}) = -i\epsilon^{-\frac{1}{2}}(\text{ber}' x + i \text{bei}' x),$$

$$J_1'(x\epsilon^{\frac{1}{2}}) = -i(\text{ber}'' x + i \text{bei}'' x),$$

and

$$\frac{\mu}{ka} \cdot \frac{J_1}{J_1'} = \frac{\mu}{x} \cdot \frac{\text{ber}' x + i \text{bei}' x}{\text{ber}'' x + i \text{bei}'' x},$$

whence on reduction

$$(1 - \mu J_1/ka J_1')/(1 + \mu J_1/ka J_1') = (E + iF)/D, \quad (6)$$

where

$$\left. \begin{aligned} Ex^2 &= (x \text{ber}'' x)^2 + (x \text{bei}'' x)^2 - (\mu \text{ber}' x)^2 - (\mu \text{bei}' x)^2, \\ Fx &= 2\mu(\text{ber}' x \text{bei}'' x - \text{bei}' x \text{ber}'' x), \\ Dx^2 &= (x \text{ber}'' x)^2 + (x \text{bei}'' x)^2 + (\mu \text{ber}' x)^2 + (\mu \text{bei}' x)^2 \\ &\quad + 2\mu x(\text{ber}' x \text{ber}'' x + \text{bei}' x \text{bei}'' x). \end{aligned} \right\} \quad (7)$$

But if accents denote differentiations with respect to x ,

$$y = \text{ber } x + i \text{ber } x$$

is a solution of

$$xy'' + y' = \iota xy,$$

so that

$$x \text{ber}'' x + \iota x \text{bei}'' x + \text{ber}' x + \iota \text{bei}' x = \iota x (\text{ber } x + \iota \text{bei } x),$$

and therefore

$$\left. \begin{aligned} x \text{ber}'' x &= -\text{ber}' x - x \text{bei } x, \\ x \text{bei}'' x &= x \text{ber } x - \text{bei}' x, \end{aligned} \right\} \dots \dots (8)$$

from which may be deduced

$$\left. \begin{aligned} \text{ber}' x \text{bei}'' x - \text{bei}' x \text{ber}'' x &= \text{ber } x \text{ber}' x + \text{bei } x \text{bei}' x, \\ \text{ber}' x \text{ber}'' x + \text{bei}' x \text{bei}'' x &= \text{ber } x \text{bei}' x - \text{bei } x \text{ber}' x \\ &\quad - x^{-1}(\text{ber}' x)^2 - x^{-1}(\text{bei}' x)^2, \\ (x \text{ber}'' x)^2 + (x \text{bei}'' x)^2 &= (\text{ber}' x)^2 + (\text{bei}' x)^2 + (x \text{ber } x)^2 \\ &\quad + (x \text{bei } x)^2 - 2x(\text{ber } x \text{bei}' x - \text{bei } x \text{ber}' x). \end{aligned} \right\} \dots (9)$$

Writing

$$\left. \begin{aligned} P &= \text{ber}^2 x + \text{bei}^2 x, \\ \alpha P &= (\text{ber}' x)^2 + (\text{bei}' x)^2, \\ \beta P &= \text{ber } x \text{ber}' x + \text{bei } x \text{bei}' x, \\ \gamma P &= \text{ber } x \text{bei}' x - \text{bei } x \text{ber}' x. \end{aligned} \right\} \dots \dots (10)$$

Then

$$Ex^2/P = \alpha(1 - \mu^2) + x^2 - 2\gamma x,$$

$$Fx/P = 2\mu\beta,$$

$$Dx^2/P = \alpha(1 - \mu)^2 + x^2 - 2x\gamma(1 - \mu),$$

where

$$D\lambda_1 = E + \iota F. \dots \dots (11)$$

For the special case $\mu = 1$; $D = P$, so that

$$\lambda_1 = 1 - 2(\gamma - \iota\beta)/x, \dots \dots (12)$$

a very simple result, which is true in practice for all but iron wires. Again, on reduction,

$$\log a/c - \mu J_0/kaJ_0' = \log a/c - \mu(\beta - \iota\gamma)/\alpha x,$$

$$\log ha - \mu J_0/haJ_0' = \log ha - \mu(\beta - \iota\gamma)/\alpha x,$$

and their quotient becomes

$$(A + \iota B)/C,$$

where

$$\left. \begin{aligned} x^2 A &= \alpha x^2 \log ha \cdot \log a/c - \mu \beta x \log ha^2/c + \mu^2, \\ xB &= \mu \gamma \log hc, \\ x^2 C &= \alpha x^2 \log^2 ha - 2\mu \beta x \log ha + \mu^2, \end{aligned} \right\} \quad (13)$$

and it has been noticed that $\beta^2 + \gamma^2 = \alpha$ identically.

With these values, it is found that, M and N being given by (3),

$$c^2 CD(M + iN) = 4a^2(E + iF)(A + iB).$$

$$\text{Thus} \quad \left. \begin{aligned} M &= 4a^2(AE - BF)/c^2 CD, \\ N &= 4a^2(AF + BE)/c^2 CD. \end{aligned} \right\} \quad (14)$$

Moreover,

$$\frac{CD}{P} = \left(\alpha \log^2 ha - \frac{2\mu\beta}{x} \log ha + \frac{\mu^2}{x^2} \right) \left(\frac{\alpha}{x^2} \frac{1}{1-\mu^2} + 1 - \frac{2\gamma}{x} \frac{1}{1-\mu} \right). \quad (15)$$

$$\begin{aligned} \frac{AE - BF}{P} &= \left(\alpha \log \frac{a}{c} \log ha - \frac{\mu\beta}{x} \log \frac{ha^2}{c} + \frac{\mu^2}{x^2} \right) \left(\frac{\alpha - \alpha\mu^2}{x^2} \right. \\ &\quad \left. + 1 - \frac{2\gamma}{x} \right) - \frac{2\mu^2\beta\gamma}{x^2} \log hc. \quad (16) \end{aligned}$$

The succeeding reduction depends upon the particular circumstances of the wires. We now write $\mu=1$, and suppose that the wires are of copper. The only case needing separate treatment is that of iron, where μ is very large.

Inductance when $\mu=1$. High frequency.

Quoting the value of M in (14), the inductance becomes, when $\mu=1$,

$$L = 4 \log_e \frac{c}{a} + \frac{4\beta}{\alpha x} + \frac{4a^2(AE - BF)}{c^2 CD}, \quad (17)$$

where

$$\left. \begin{aligned} \frac{AE - BF}{P} &= \left(\alpha \log \frac{a}{c} \log ha - \frac{\beta}{x} \log \frac{ha^2}{c} + \frac{1}{x^2} \right) \\ &\quad \left(1 - \frac{2\gamma}{x} \right) - \frac{2\beta\gamma}{x^2} \log hc, \\ \frac{CD}{P} &= \alpha \log^2 ha - \frac{2\beta}{x} \log ha + \frac{1}{x^2}. \end{aligned} \right\} \quad (18)$$

Asymptotic formulæ for the functions (α, β, γ) have been developed by Dr. A. Russell*.

* Proc. Phys. Soc. vol. xxii.

When the argument x is not less than 8, then to four significant figures, if $\lambda = x\sqrt{2}^*$, so that, in terms of the characteristics of the two wires,

$$\lambda = 2a \left(\frac{2\pi n}{\sigma} \right)^{\frac{1}{2}}, \dots \dots \dots (19)$$

we may write

$$\left. \begin{aligned} \alpha &= 1 - \frac{1}{\lambda} + \frac{1}{2\lambda^2} + \frac{3}{4\lambda^3}, \\ \beta \sqrt{2} &= 1 - \frac{1}{\lambda} - \frac{1}{4\lambda^2}, \\ \gamma \sqrt{2} &= 1 + \frac{1}{4\lambda^2} + \frac{1}{2\lambda^3}, \end{aligned} \right\} \dots \dots \dots (20)$$

and after reduction

$$\left. \begin{aligned} \frac{AE-BF}{P} &= \left(1 - \frac{3}{\lambda} + \frac{5}{2\lambda^2} - \frac{3}{4\lambda^3} \right) \log \frac{a}{c} \log ha \\ &\quad - \frac{2}{\lambda} \left(1 - \frac{3}{\lambda} - \frac{1}{4\lambda^2} \right) \log ha + \frac{1}{\lambda} \left(1 - \frac{5}{\lambda} \right. \\ &\quad \left. + \frac{15}{4\lambda^2} \right) \log hc + \frac{2}{\lambda^2} \left(1 - \frac{2}{\lambda} \right), \\ \frac{CD}{P} &= \left(1 - \frac{1}{\lambda} + \frac{1}{2\lambda^2} + \frac{3}{4\lambda^3} \right) \log^2 ha - \frac{2}{\lambda} \left(1 - \right. \\ &\quad \left. \frac{1}{\lambda} - \frac{1}{4\lambda^2} \right) \log ha + \frac{2}{\lambda^2}. \end{aligned} \right\} \dots \dots \dots (21)$$

We proceed to examine the limiting frequency for which the four-figure accuracy of these results is preserved. For a copper wire of low resistance as ordinarily used, the resistivity may be taken as 1696 C.G.S. If f be the frequency of alternation, $n = 2\pi f$, leading to $x = \frac{1}{5} a f^{\frac{1}{2}}$ approximately.

Thus $x = 8$ leads to the determining relation

$$a f^{\frac{1}{2}} = 40. \dots \dots \dots (22)$$

If $a f^{\frac{1}{2}}$ is less than 40, the four-figure accuracy, in so far as it depends on x , is lost. When f is only 400 per second, the limiting diameter of a wire becomes as large as 4 centimetres, so that the formulæ are unsuited for practice until

* This modification is more convenient for purposes of printing.

the frequency is really great. The usual range of diameter for wires applied to such purposes as telephonic communication is from 1 to 3 millimetres in the case of cables, and to 5·7 millimetres in that of overhead wires. The frequency in cases of transmission of speech will not be greater than 2000. Taking $f=1600$ as perhaps the real maximum, the limiting radius of a wire becomes 1 centimetre. The results therefore cannot have a great accuracy in such a case, and the formula (43) below must be used. But it will appear that the corresponding formula for iron leads can be so employed.

The present results have a two-figure accuracy even when $x=3$, leading to a radius of 3·75 millimetres, and therefore will apply to many cases even of telephony when thicker wires are used. In such cases, their use is preferable to that of (43), in that the calculation is more rapid.

It is necessary not to overlook the other sources of error. Firstly, in the proof of (1)*, h^2a^2 was neglected in comparison with unity. In the most unfavourable case furnished by the telephone, $f=1600$, $a=3$ millimetres, leading to $h^2a^2=10^{-14}$ approximately.

Thus this source of error needs no further consideration in such a case. But in fact, the neglect of this quantity is always legitimate. For if f be the frequency, $h^2a^2=4f^2 10^{-20}$ for a wire of a centimetre radius. It can only therefore affect the fourth figure if $f=20$ million, and the second if $f=200$ million, and such a frequency is never used in combination with so thick a wire.

Secondly, a^4/c^4 was neglected in comparison with unity. Now even in ordinary telephone construction there is usually about $2\frac{1}{2}$ millimetres of paper and air between the wires, so that in the case of the cable, taking an extreme radius of 1·3 millimetres, $c=4a$, and only the third figure is affected. For overhead wires, the distance apart is about 30 centimetres.

The limiting practical closeness of the wires, when a knowledge of the self-induction is required, is, I believe, attained in Mr. A. Campbell's experiments on variable mutual inductances†. The error may then be so great as one or two per cent.

* Phil. Mag. Feb. 1909.

† Phil. Mag. Jan. 1908.

Effect of a Small Capacity.

Heaviside * has divided circuits into five classes, with the following determining properties:—

(1) Submarine cables proper, in which the capacity is the main factor, and whose treatment must follow the electrostatic theory.

(2) Short lines of low frequency, in which self-induction and resistance determine the effects. This class includes the majority of short telephone circuits.

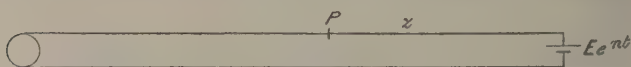
(3) A class in which capacity and self-induction are equally important. These circuits are very difficult in theory.

(4) Circuits of high frequency, but small capacity and resistance, and with the inductance so chosen that signals may travel without distortion. For this class the electrostatic theory cannot be applied however long the circuit.

(5) Circuits in which distortionless propagation is obtained by allowing an electric leakage.

The effect of leakage is small in all but the fifth case, and will be ignored.

Let an impressed force Ee^{nt} act at one end of a circuit consisting of two parallel wires, with terminal apparatus whose inductance, capacity, and resistance are neglected. Let z denote distance of a point P from the end at which the force acts. If R , L , C be the resistance, inductance,



and capacity per unit length, and (ι_1 , V) the current and potential at a point P; then

$$\left. \begin{aligned} C \partial V / \partial t &= -\partial \iota_1 / \partial z \\ L \partial \iota_1 / \partial t + R \iota_1 &= -\partial V / \partial z \end{aligned} \right\}, \quad \dots (23)$$

where $\partial / \partial t \equiv m$.

The solution of these equations is found to be

$$\iota_1 = (E/I)e^{(nt-\alpha)}, \quad \dots (24)$$

* 'Electrical Papers,' vol. ii.

where

$$I = L^{\frac{1}{2}} C^{-\frac{1}{2}} (1 + R^2/n^2 L^2)^{\frac{1}{2}} (\cosh 2lQ_1 - \cos 2lQ_2)^{\frac{1}{2}} \left. \vphantom{I = L^{\frac{1}{2}} C^{-\frac{1}{2}} (1 + R^2/n^2 L^2)^{\frac{1}{2}} (\cosh 2lQ_1 - \cos 2lQ_2)^{\frac{1}{2}}} \right\} \quad (25)$$

$$Q_1, Q_2 = n(\frac{1}{2}LC)^{\frac{1}{2}} \{ (1 + R^2/n^2 L^2)^{\frac{1}{2}} \mp 1 \}^{\frac{1}{2}}.$$

With the value of α we are not concerned. For the effect of leakage, reference may be made to Heaviside's paper*.

$$\text{If } C = 0, \quad I = l(R^2 + L^2 n^2)^{\frac{1}{2}}, \quad . \quad . \quad . \quad (26)$$

which is the ordinary impedance assumed to be valid in the previous paper. In the electrostatic theory, we write $L=0$, which is obviously unjustifiable if R is small or n very great, so that circuits of the first kind lose their character in these cases.

We proceed to estimate the correction to be made in the value of L , to take account of a small capacity, regarding only the first order term. Writing

$$\lambda_1^2, \lambda_2^2 = (1 + R^2/n^2 L^2)^{\frac{1}{2}} \mp 1,$$

then on reduction,

$$\begin{aligned} \cosh 2lQ_1 - \cos 2lQ_2 &= L C n^2 l^2 (\lambda_1^2 + \lambda_2^2) \{ 1 + \frac{1}{6} L C n^2 l^2 (\lambda_1^2 - \lambda_2^2) \} \\ &= 2 C n l^2 (R^2 + L^2 n^2)^{\frac{1}{2}} (1 - \frac{1}{3} l^2 n^2 L C), \end{aligned}$$

so that

$$I = l\sqrt{2} \cdot (R^2 + L^2 n^2)^{\frac{1}{2}} (1 - \frac{1}{6} l^2 n^2 L C).$$

If L' be the equivalent induction when this small capacity is taken into account,

$$R^2 + L'^2 n^2 = (R^2 + L^2 n^2) (1 - \frac{1}{3} l^2 n^2 L C)$$

$$\text{or } (L' - L)/L = -l^2 C (R^2 + L^2 n^2)/6L \text{ approximately.} \quad (27)$$

This equation serves to limit the types of circuit to which the uncorrected inductance formulæ may be applied, in so far as error due to capacity is concerned. For example, if a three-figure accuracy is required, it is necessary that

$$l^2 C (R^2 + L^2 n^2)/6L \nless 10^{-4}, \quad . \quad . \quad . \quad (28)$$

where l is the length of either wire of the circuit.

* *Loc. cit.*

We have also neglected

$$\frac{1}{90} l^4 n^4 L^2 C^2 (\lambda_1^4 + \lambda_2^4 - \lambda_1^2 \lambda_2^2) = \frac{1}{90} l^4 n^4 L^2 C^2 (4 + R^2 / l^2 n^2)$$

in comparison with unity. This restricts the capacity for which the correction (27) may be used. Thus, for low resistance,

$$4l^4 n^2 C^2 \gtrsim 9 \cdot 10^{-5}.$$

This equation will ordinarily supply the lower limit for C . Thus for frequencies such that

$$nC \gtrsim (5l^2)^{-1} 10^{-2} \quad . \quad . \quad . \quad (29)$$

the uncorrected (for capacity) formulæ may be used provided that C is also so small as to make

$$Cn^2 L \gtrsim 6l^{-2} 10^{-4}, \quad . \quad . \quad . \quad (30)$$

again neglecting R .

For a capacity of one microfarad per kilometre, $C = 10^{-20}$. In a case like that of the Atlantic cable the capacity is about a quarter of this amount, and the limiting frequency, with l in centimetres, is about $l^{-2} 10^{18}$. This excludes such a cable altogether from the investigation, although a shorter cable of the same radius of wire and capacity gradient can be within its scope. For example, a frequency of so much as 10 million can be treated if l is not greater than 1.2 kilometres, for a suitable range of inductance given by (30). The range of inductance is of course dependent mainly on the distance between the wires.

Obviously all short telephone circuits satisfy both (29) and (30), and their capacity needs no consideration.

Copper Wires with High Frequency.

With the above restrictions as regards capacity and frequency (the latter being of little practical import), writing in (17),

$$\begin{aligned} \frac{\beta}{\alpha x} &= \frac{1}{\lambda} \left(1 - \frac{1}{\lambda} - \frac{1}{4\lambda^2} \right) \left(1 - \frac{1}{\lambda} + \frac{1}{2\lambda^2} \right) \\ &= \frac{1}{\lambda} \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right). \quad . \quad . \quad . \quad (31) \end{aligned}$$

Then with the values of (21),

$$L = 4 \log_e \frac{c}{a} + \frac{4}{\lambda} \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right) + \frac{4a^2}{c^2 CD} (AE - BF),$$

where $\lambda = 2a(2\pi\mu n/\sigma)^{\frac{1}{2}} = 4\pi a(8/\sigma)^{\frac{1}{2}},$

with a four-figure accuracy so far as λ is concerned, if $\lambda > 8\sqrt{2}$. But a further simplification may be introduced. Since ha is a very small magnitude in actual cases, its logarithm is large and negative. Thus if $\rho = \log_e ha$, the functions may be expanded in a descending series of powers of ρ , and

$$\frac{CD}{P\rho^2} = 1 - \frac{1}{\lambda} \left(1 + \frac{2}{\rho} \right) + \frac{1}{\lambda^2} \left(\frac{1}{2} + \frac{2}{\rho} + \frac{2}{\rho^2} \right) + \frac{1}{\lambda^3} \left(\frac{3}{4} + \frac{1}{2\rho} \right)$$

or

$$\frac{P\rho^2}{CD} = 1 + \frac{1}{\lambda} \left(1 + \frac{2}{\rho} \right) + \frac{1}{\lambda^2} \left(\frac{1}{2} + \frac{2}{\rho} + \frac{2}{\rho^2} \right) - \frac{1}{\lambda^3} \left(\frac{3}{4} + \frac{1}{2\rho} \right), \quad (32)$$

whence, after considerable reduction

$$\begin{aligned} \frac{\rho^2}{CD} (AE - BF) &= \rho \log \frac{a}{c} - \frac{1}{\lambda} \left(\rho + 4\rho \log \frac{a}{c} + 3 \log \frac{a}{c} \right) \\ &\quad + \frac{1}{4\rho\lambda^3} \left\{ 8\rho - 8\rho^2 + 19\rho^3 - (8 - 8\rho + 3\rho^2) \log \frac{a}{c} \right\}. \end{aligned}$$

The vanishing of the coefficient of λ^{-2} is curious.

Finally, for a pair of copper wires, the main error being of relative magnitude a^4/c^4 when they are close together, if $\lambda < 8\sqrt{2}$,

$$\lambda = 4\pi a (f/\sigma)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (33)$$

if f is the frequency, then

$$\begin{aligned} L &= 4 \log_e \frac{c}{a} + \frac{4}{\lambda} \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right) + \frac{4a^2}{c^2} \log \frac{a}{c} \\ &\quad - \frac{4a^2}{c^2\lambda\rho^2} \left(\rho + 4\rho \log \frac{a}{c} + 3 \log \frac{a}{c} \right) \\ &\quad + \frac{a^2}{c^2\lambda^3\rho^4} \left\{ 8\rho - 8\rho^2 + 19\rho^3 - (8 - 8\rho + 3\rho^2) \log \frac{a}{c} \right\}, \quad (34) \end{aligned}$$

where $\rho = \log_e ha$, and the limiting capacity is given by (29). For most useful purposes it may be greatly shortened, according to the value of λ .

Iron Wires with High Frequency.

When iron is used, there is no object to be served in obtaining so accurate a formula, on account of the vagaries in the magnetic behaviour of the metal. But the approximate value of μ is known, for Lord Rayleigh * has shown that for feeble magnetizing forces of periodic character, μ usually lies between 90 and 100, and becomes much greater as the forces are increased. It is therefore lawful to take μ as of order 10^2 . In the formulæ (20), (α, β, γ) never become large, and therefore whatever the value of x (or λ), the approximation may be conducted by expanding the functions in inverse powers of μ .

Thus if $\rho = \log ha$ as before,

$$\begin{aligned} \frac{CD}{P} &= \frac{\alpha\mu^4}{x^4} \left(1 - 2\beta\rho \frac{x}{\mu} + \alpha\rho^2 \frac{x^2}{\mu^2} \right) \left(1 - \frac{2}{\mu} \left(1 - \frac{\gamma x}{\alpha} \right) + \frac{1}{\mu^2} \left(1 - \frac{2\gamma x}{\alpha} + \frac{x^2}{\alpha^2} \right) \right) \\ &= \frac{\mu^4 \alpha}{x^4} \left\{ 1 - \frac{2}{\mu} \left(\beta\rho x + 1 - \frac{\gamma x}{\alpha} \right) + \frac{1}{\mu^2} \left(1 + \alpha\rho^2 x^2 - \frac{2\gamma x}{\alpha} \right. \right. \\ &\quad \left. \left. + \frac{x^2}{\alpha} + 4\beta\rho x - 4\beta\gamma\rho \frac{x^2}{\alpha} \right) \right\} \\ \text{or} \\ \frac{P\mu^4 \alpha}{x^4 CD} &= 1 + \frac{2}{\mu} \left(1 + \beta\rho x - \frac{\gamma x}{\alpha} \right) + \frac{1}{\mu^2} \left(3 + 4\beta^2\rho^2 x^2 + 4\beta\rho x + \frac{4\gamma^2 x^2}{\alpha^2} \right. \\ &\quad \left. - 4\beta\gamma\rho \frac{x^2}{\alpha} - \frac{x^2}{\alpha} - \frac{6\gamma x}{\alpha} - \alpha\rho^2 x^2 \right). \quad (35) \end{aligned}$$

This neglects the ratio x^3/μ^3 . Taking a wire of gauge $\frac{1}{2}$ cm., an approximate practical limit for overhead wires, and writing, as an ordinary value for iron, $\sigma = 10,000$ c.g.s.,

$$x = \frac{3}{20} f^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (36)$$

if $\mu = 100$. Thus even when $f = 10,000$ per second, neglect of x^3/μ^3 only affects the third figure of CD , and therefore at most the fourth of L , when the ratio a^2/c^2 is noticed. This process is justifiable whether the ascending or descending series are used for (α, β, γ) .

* Scientific Papers, iii.

In a similar manner,

$$\frac{(AE-BF)}{P} = + \frac{\alpha\mu^4}{x^4} \left(1 - \frac{\beta x}{\mu} \log \frac{ha^2}{c} + \frac{\alpha x^2}{\mu^2} \log ha \log \frac{a}{c} \right) \left(1 - \frac{\alpha + x^2 - 2\gamma x}{\alpha\mu^2} \right) \\ + 2\mu^2 \frac{\beta\gamma}{x^2} \log hc;$$

and after some reduction,

$$-\frac{AE-BF}{(D)} = 1 + \frac{1}{\mu} \left(2 + \beta\rho x - \frac{2\gamma x}{\alpha} - \beta x \log \frac{a}{c} \right) \\ + \frac{1}{\mu^2} \left(2 + 2\beta\rho x - \alpha\rho^2 x^2 + 2\beta^2\rho^2 x^2 + \frac{4\gamma^2 x^2}{\alpha^2} - \frac{4\gamma x}{\alpha} - \frac{2x^2}{\alpha^2} \right) \\ + \frac{1}{\mu^2} \log \frac{a}{c} \cdot (\alpha x^2 \rho - 2\beta^2 x^2 \rho - 2\beta x).$$

In the extreme case above, $\frac{x}{\mu^2} \nless \frac{1}{600}$, and the final error consequent upon its neglect is of order $(6000)^{-1}$ relative to unity even when the wires have limiting practical closeness.

Accordingly,

$$-\left(\frac{AE-BF}{(D)} \right) = 1 + \frac{1}{\mu} \left(2 + \beta\rho x - \frac{2\gamma x}{\alpha} - \beta x \log \frac{a}{c} \right) \\ + \frac{1}{\mu^2} \left(2\beta^2\rho^2 x^2 - \alpha\rho^2 x^2 + \frac{4\gamma^2}{\alpha^2} x^2 - \frac{2x^2}{\alpha} + x^2\rho(\alpha - 2\beta^2) \log \frac{a}{c} \right).$$

Consider now the case in which $x > 8$. In the coefficient of μ^{-2} , the first order values of $(\alpha\beta\gamma)$ may be used, with the second order in that of μ^{-1} .

Under these conditions, the coefficient of μ^{-2} is found to vanish, leading to the very simple result, to order μ^{-3} ,

$$-\left(\frac{AE-BF}{(D)} \right) = 1 + \frac{1}{\mu} \left\{ 2 + \frac{1}{2}\lambda \left(1 - \frac{1}{\lambda} \right) \log hc - \lambda \left(1 + \frac{1}{\lambda} \right) \right\} \\ = 1 + (\lambda - 1)(\log hc - 2)/2\mu,$$

and thus for iron wires with $\lambda > 8\sqrt{2}$,

$$L = 4 \log_e \frac{c}{a} + \frac{4\mu}{\lambda} \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right) - \frac{4a^2}{c^2} - \frac{2a^2}{c^2\mu} (\lambda - 1)(\log hc - 2). \quad (37')$$

where

$$\lambda = 4\pi a \left(\frac{f\mu}{\sigma} \right)^{\frac{1}{2}}, \text{ and } \frac{a^2}{c^2} \frac{\lambda}{\mu^2} \text{ is neglected.}$$

On account of μ , this formula may be applied, for any given frequency, to a wire of only about a quarter the corresponding limiting radius in the case of copper. For example, for a frequency 800, and radius 2 millimetres, the value of x becomes 5, with $\mu=100$. Thus a three-figure accuracy is obtained for the most unfavourable case of the telephone circuit. Accordingly, for a telephone circuit of iron wires not twisted together, this result is of universal application.

Iron Wires with Small Frequency.

This case is not very important, but the result may be given at once. Thus if x is not greater than 2, we may neglect the term involving μ^{-2} , and write, to three significant figures,

$$L = 4 \log \frac{c}{a} + \frac{4\mu\beta}{x\alpha} - \frac{4a^2}{c^2} - \frac{4a^2}{c^2\mu} \left(\beta x \log hc + 2 - \frac{2\gamma x}{\alpha} \right). \quad (38)$$

The values of (α, β, γ) suited to this case have been given by Russell*.

$$\begin{aligned} \text{If } z = \frac{x}{2} = a\pi \left(\frac{2\mu f}{\sigma} \right)^{\frac{1}{2}}, \\ \left. \begin{aligned} \alpha &= z^2 \left(1 - \frac{5}{12} z^4 + \frac{143}{720} z^8 - \frac{7661}{4^2 \cdot 7!} z^{12} \right) \\ \beta &= \frac{1}{2} z^3 \left(1 - \frac{11}{24} z^4 + \frac{473}{3 \cdot 6!} z^8 - \frac{304107}{4 \cdot 12^2 \cdot 7!} z^{12} \right) \\ \gamma &= z \left(1 - \frac{1}{3} z^4 + \frac{19}{5!} z^8 - \frac{687}{7 \cdot 6^4} z^{12} \right) \end{aligned} \right\}. \quad (39) \end{aligned}$$

Moreover,

$$\left. \begin{aligned} \frac{\beta}{\alpha} &= \frac{1}{2} z \left(1 - \frac{1}{24} z^4 + \frac{13}{4320} z^8 - \frac{647}{12^2 \cdot 360 \cdot 56} z^{12} \right) \\ \frac{\gamma}{\alpha} &= z^{-1} \left(1 + \frac{1}{12} z^4 - \frac{1}{180} z^8 + \frac{11}{12 \cdot 28 \cdot 30} z^{12} \right) \end{aligned} \right\} \quad (40)$$

* *Loc. cit.*

Heaviside* has also given the value of β/α , and Lord Rayleigh† that of γ/α , by different modes of proof. Thus finally, when

$$z = \alpha\pi(2\mu f/\sigma)^{\frac{1}{2}} \gg 1, \quad . \quad . \quad . \quad . \quad (41)$$

$$\begin{aligned} L = 4 \log \frac{c}{a} + \mu \left(1 - \frac{1}{24} z^4 + \frac{13}{4320} z^8 - \frac{647}{12^2 \cdot 360 \cdot 56} z^{12} \right) - \frac{4a^2}{c^2} \\ - \frac{4z^4 a^2}{c^2 \mu} \log_e hc \left\{ 1 - \frac{11}{24} z^4 + \frac{473}{3 \cdot 6!} z^8 - \frac{304107}{4 \cdot 12^2 \cdot 7!} z^{12} \right\} \\ + \frac{8a^2}{c^2 \mu} \left(1 + \frac{1}{6} z^4 + \frac{1}{90} z^8 - \frac{11}{28 \cdot 180} z^{12} \right). \quad . \quad . \quad . \quad . \quad (42) \end{aligned}$$

Small Frequency, Copper Wires.

In the notation of (18), with $\rho = \log_e ha$,

$$x^2 \frac{CD}{P} = 1 - 2\beta x \rho + \alpha x^2 \rho^2.$$

The frequencies for which this formula will ordinarily be required lie between 200 and 1600, and the radii between 1 and 3 millimetres. These values cause ρ to vary between about 13 and 16 only, and z between $\frac{1}{4}$ and unity, so that P/CD cannot be expanded in ascending or descending powers of ρ or z in general. The result may be most simply written in the form

$$\begin{aligned} L = 4 \log \frac{c}{a} + \frac{2}{z} \frac{\beta}{\alpha} + \frac{4a^2}{c^2} \left(1 - 2\beta \rho z - \frac{\gamma}{z} \right) / (1 - 4\beta \rho z + 4\alpha \rho^2 z^2) \\ + \frac{8a^2}{c^2} \log \frac{a}{c} \cdot (2\alpha \rho z^2 - 2\alpha \rho \gamma z - \beta z + 2\beta \gamma) / (1 - 4\beta \rho z + 4\alpha \rho^2 z^2), \quad (43) \end{aligned}$$

where $(\alpha \beta \gamma z)$ are given by (39), and $\rho = \log_e ha$.

The two final terms in the last numerator are only about $\frac{1}{50}$ of the other two, and may be ignored.

* Elec. Papers, ii. p. 64.

† Phil. Mag. xxii. p. 381 (1886); Scientific Papers, vol. ii.

II. EFFECTIVE RESISTANCE.

Copper Wires with High Frequency.

The limitations of all the formulæ below, except when otherwise stated, are the same as in the corresponding inductance formulæ. With the previous notation, the effective resistance per unit length of the two wires is

$$R = \frac{4n\mu}{x} \frac{\text{ber } x \text{ bei}' x - \text{bei } x \text{ ber}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} - nN$$

$$= \frac{4n\mu}{x} \cdot \frac{\gamma}{\alpha} - \frac{4a^2n}{c^2 CD} (AF + BE). \quad (44)$$

where CD/P has the value in (15), and on reduction

$$\frac{AF + BE}{P} = \frac{2\mu\beta}{x} \left(\alpha \log ha \log \frac{a}{c} - \frac{\mu\beta}{x} \log \frac{ha^2}{c} + \frac{\mu^2}{x^2} \right)$$

$$+ \frac{\mu\gamma}{x} \log hc \left(1 - \frac{2\gamma}{x} + \frac{\alpha}{x^2} \overline{1 - \mu^2} \right). \quad (45)$$

When $\mu = 1$, and the frequency is high, so that, if $x\sqrt{2} = \lambda$, the formulæ (20) may be employed, then after further reduction,

$$\frac{AF + BE}{P} = \frac{2}{\lambda} \log_c^a \log ha \left(1 - \frac{2}{\lambda} + \frac{5}{4\lambda^2} \right) - \frac{4}{\lambda^2} \log hu \left(1 - \frac{2}{\lambda} \right)$$

$$+ \frac{1}{\lambda} \log hc \left(1 - \frac{15}{4\lambda^2} \right) + \frac{4}{\lambda^3},$$

and with the help of (32), we ultimately obtain

$$\frac{\lambda\rho^2}{CD} (AF + BE) = 2\rho \log_c^a + \rho - \log_c^a - \frac{1}{\lambda\rho} \left\{ \rho^2 + \log_c^a (2 - 3\rho + 2\rho^2) \right\}$$

$$+ \frac{1}{4\lambda^2\rho^2} \left\{ 8\rho - 8\rho^2 + 3\rho^3 - \log_c^a (8 - 8\rho + 3\rho^2 + 2\rho^3) \right\}$$

where $\rho = \log_e ha$.

Finally, for a pair of copper wires, with high frequency, and with the limitations of (34),

$$R = \frac{4n}{\lambda} \left(1 + \frac{1}{\lambda} + \frac{3}{4\lambda^2} \right) - \frac{4a^2n}{\lambda^2 c^2 \rho^2} \left\{ \rho + (2\rho - 1) \log_c^a \right\}$$

$$+ \frac{4a^2n}{\lambda^2 c^2 \rho^3} \left\{ \rho^2 + (2 - 3\rho + 2\rho^2) \log_c^a \right\}$$

$$- \frac{a^2n}{\lambda^2 c^2 \rho^4} \left\{ 8\rho - 8\rho^2 + 3\rho^3 - (8 - 8\rho + 3\rho^2 + 2\rho^3) \log_c^a \right\}. \quad (46)$$

Iron Wires with High Frequency.

When μ is large, we may write from (45),

$$\frac{AF+BE}{P} = \frac{\mu^3}{x^3} \left(2\beta - \alpha\gamma \log hc - 2\beta^2 \frac{x}{\mu} \log \frac{ha^2}{c} \right), \quad (47)$$

and by (35),

$$\frac{P\mu^4\alpha}{x^4(D)} = 1 + \frac{2}{\mu} \left(1 + \beta\rho x - \gamma \frac{x}{\alpha} \right),$$

so that

$$\frac{AF+BE}{CD} = \left(\frac{2\beta}{\alpha} - \gamma \log hc \right) - \frac{2x}{\alpha\mu^2} \left\{ \left(1 + \beta\rho x - \gamma \frac{x}{\alpha} \right) (2\beta - \alpha\gamma \log hc) + \beta^2 x \log \frac{ha^2}{c} \right\}.$$

Neglecting x/μ^2 as before, and writing the first approximations to $(\alpha\beta\gamma)$ in the second term, and the second in the first, we finally obtain

$$\frac{AF+BE}{(D)} = \frac{\lambda}{2\mu} \left(2 - \rho + \log \frac{a}{c} \right) - \frac{\lambda^2}{2\mu^2} \left(4\rho - 2 - \rho^2 + \rho \log \frac{a}{c} \right),$$

the term in $\lambda^0\mu^{-1}$ vanishing identically.

Thus the effective resistance becomes

$$R = \frac{4n\mu}{\lambda} \left(1 + \frac{1}{\lambda} + \frac{3}{4\lambda^2} \right) - \frac{2a^2n\lambda}{c^2\mu} \left(2 - \rho + \log \frac{a}{c} \right) + \frac{2a^2n\lambda^2}{c^2\mu^2} \left(4\rho - 2 - \rho^2 + \rho \log \frac{a}{c} \right). \quad (48)$$

This result is much more accurate than the corresponding formula (37) for inductance, owing to the presence of μ in the numerator of the first term. Moreover, λ here denotes the previous λ multiplied by $\mu^{\frac{1}{2}}$. It is applicable to all iron circuits of importance in practice, for a frequency greater than about a hundred with a radius of 1 millimetre.

Iron Wires with Low Frequency.

This result has little interest, but may be obtained at once. For $(\alpha\beta\gamma)$ are all comparable with unity in this case, and x is not greater than 2. We therefore ignore x^2/μ^2 , and obtain

$$\frac{AF+BE}{(D)} = \frac{x}{\mu} \left(\frac{2\beta}{\alpha} - \gamma \log hc \right). \quad (49)$$

This term also may be ignored unless a very high order of accuracy be desired, which can rarely happen owing to the uncertainty as to the value of μ . But if it be retained, the resistance becomes

$$R = \frac{2n\mu}{z^2} \left(1 - \frac{1}{12} z^4 - \frac{1}{180} z^8 \right) - \frac{8na^2z^2}{\mu c^2} \left(1 - \frac{1}{24} z^4 + \frac{13}{4320} z^8 \right) \\ + \frac{8na^2z^2}{\mu c^2} \left(\rho - \log \frac{a}{c} \right) \left(1 - \frac{1}{3} z^4 + \frac{19}{120} z^8 \right). \quad (50)$$

where z has the value in (39), and the brackets may be shortened except when z is nearly unity.

When the frequency is smaller, this makes $R = 2\sigma/\pi a^2$, the value appropriate to steady currents.

Copper Wires with Low Frequency.

The formula suited to this case is, writing $\beta^2 + \gamma^2 = \alpha$, if $(z \alpha \beta \gamma)$ are defined as in (39), and under the same limitations,

$$R = \frac{2n}{z} \cdot \frac{\gamma}{\alpha} - \frac{4na^2}{zc^2} (\beta - 2\alpha\rho z + 2\gamma\rho z^2)/(1 - 4\beta\rho z + 4\alpha\rho^2 z^2) \\ - \frac{8na^2}{c^2} \log \frac{a}{c} \left(\gamma^2 - \beta^2 - \gamma z + 2\alpha\beta\rho z \right)/(1 - 4\beta\rho z + 4\alpha\rho^2 z^2), \quad (51)$$

again reducing to $2\sigma/\pi a^2$ for steady currents.

Trinity College, Cambridge,
April 21, 1909.

LVIII. *The Proposed International Unit of Candle Power.*

By CLIFFORD C. PATERSON *.

(From the National Physical Laboratory.)

THE intercomparison of light units between the National or other standardizing Laboratories of America, France, Germany, and Great Britain has been proceeding at intervals since 1905.

The values which have been obtained for the ratios between the different units are now found to be in sufficiently close accord to warrant the establishment of a working basis of agreement between this country, America, and France in the matter of a common unit of Candle Power.

The writer has been conducting the photometric measurements connected with the work, in this country, and it is the intention in this paper to give the results of the comparisons which have been made, and to explain briefly those facts connected with the different standards concerned which have a bearing on the agreement which has been reached.

The possibility of agreement between the British and French units was demonstrated by Dr. Glazebrook in a paper on light standards read before the B.A. at Dublin in 1908 †. The chief factor in the present movement has been the desire of the authorities in the United States to establish one unit for both Gas and Electrical industries in that country, and the possibility of their adopting a unit which should be identical with those existing in Great Britain and France led them to take the initiative in an attempt to obtain international cooperation. The agreement which has resulted has the approval of the Metropolitan Gas Referees and now forms the subject of an announcement which is reproduced on page 877.

It is not necessary for the purpose of this paper to enter into a detailed description of the various standard lamps and

* Read June 11, 1909.

† B.A. Report, 1908, Dublin, "The Photometric Standard of the National Physical Laboratory"; also 'Journal of Gas Lighting,' vol. 103, 1908, p. 713.

units referred to in the memorandum. Dr. J. A. Fleming discussed the question of light standards very fully in his paper before the Inst. of Elect. Engineers, vol. 32, to which reference should be made*. Some facts, however, connected with the units in question have a more especial bearing on the experimental results, and should be borne in mind in connexion with the table of ratios given on pages 874, 875.

The British Unit.—As defined in the above-mentioned recommendation, the unit of Candle Power in this country is the Harcourt 10 C.P. Pentane Lamp burning in an atmosphere at normal barometric pressure and containing 8 litres of water-vapour per cubic metre as measured by a ventilated hygrometer.

A difficulty arises in the use of all flame standards in connexion with the method to be employed for measuring the humidity. When flame lamps are burning in a closed room it is well known that their candle power diminishes, due probably to the vitiation of the air in the immediate neighbourhood of the flame†. Two standards will not necessarily diminish in Candle Power at the same rate, and it is therefore necessary to take readings after the air of the room has been changed and before the C.P. of the lamps has had time to be affected. The method of measuring humidity must therefore be a rapid one, and it is now generally agreed that from considerations of accuracy and quickness of reading the ventilated hygrometer is the best instrument to use‡. In the German and French comparisons this has been used, but in the English comparisons (as reported to the Photometric Commission meeting in Zurich in 1907) the unventilated hygrometer was employed§, and in the tables, published in the proceedings of the Commission the author's

* See also paper by the author, Journ. Inst. Elect. Eng. vol. xxxviii. p. 271.

† Report Amer. Gas Inst., "Methods of taking C.P. of Gas," Illum. Eng. 1909, p. 203.

‡ Proc. Roy. Soc. Edinburgh, vol. xliii., 1905; also "Zur kenntniss des Ventilierten Psychrometers," Akademische Abhandlung der Fakultät der Universität zu Upsala, by Aron Svensson, 1898.

§ See B.A. Report, Dublin 1908.

results are given in terms of humidity measured by this instrument. The ratios tabulated in the present paper are corrected so as to be in terms of the ventilated hygrometer. In each case values are taken for the humidity at which each lamp, in the country where it forms the standard, is considered to give its nominal Candle Power.

The Unit of the United States of America.—In the initial adoption of a unit of Candle Power the United States of America endeavoured to make its value as nearly as possible the same as that accepted at the time in this country*. This was before Prof. Vernon Harcourt and the Metropolitan Gas Reterees (London) had established the 10 Candle Pentane Lamp on the present definite basis.

The American Inst. of Electrical Engineers recommended the derivation of their unit from the Hefner Lamp by increasing its value in the ratio of 0.88 to 1. This seemed at the time, from different observers' work, to be the most probable ratio between the Hefner and British units. The gas industry in America, however, did not follow this course but developed their unit along the lines of the 10 Candle Pentane Lamp†. The result is that there has been, up to now, an appreciable difference between the units adopted in the two industries in that country. The Illuminating Engineering Society and other bodies took the matter up energetically, and the Bureau of Standards, Washington, now has the support of the leading institutions in America, in defining the value of a common standard, to be accepted throughout the States. This Institution has ascertained by means of electric intercomparisons the ratio of their present unit to those of Germany, France, and Great Britain respectively‡, and has arranged to adjust the value of the American unit as already indicated.

German Unit.—The unit accepted in Germany is the light given by the Hefner Lamp burning in an atmosphere at

* Bulletin of the Bureau of Standards, vol. iii., no. 1, p. 65; Report to the American Gas Institute on "A Unit of Light," Journal of Gas Lighting, vol. 104, 1908, p. 426.

† "The Working Standards of Light and their Use in the Photometry of Gas," Ch. O. Bond, Franklin Inst. 1908.

‡ Ref. cit.

normal barometric pressure and containing 8·8 litres of water vapour per cubic metre. The researches of Liebhenthal * at the Reichsanstalt on the Hefner Lamp and the variation of its C.P. with atmospheric change were the earliest systematic experiments undertaken of this nature and are too well known to require more than passing mention.

French Unit.—The Candle Power Unit adopted by the Electrical Industry in France is the Bougie Decimale. This is the 20th part of the light given out by a sq. cm. of platinum at the temperature of solidification. The unit was suggested by M. Violle and adopted by the Congrès International des Electriciens in 1881.

This standard has been found very difficult to reproduce and the French authorities still use the Carcel lamp, burning colza oil, as the standard for all photometric work.

A determination of the value of the Carcel lamp in terms of the Violle platinum standard has only been made once. This was by M. Violle himself in 1884 †. Measurements were made by him using two or three photometric methods, and all his values except one showed the bougie decimal to be a little less than 4 per cent. greater than the Carcel unit.

A multiplying factor of 1·04 for the Carcel unit was therefore given by him, and has been adopted ever since for reducing the values in terms of one standard to those of the other. As no account was taken by M. Violle of the pressure and humidity of the atmosphere in which the Carcel lamp was burning, the accepted figure of 1·04 must be regarded as liable to a certain inaccuracy due to this cause. It should be remarked, also ‡, that no correcting factor has as yet been determined for the variations in the C.P. of the Carcel lamp due to atmospheric changes. Hence, in the table given later on in the paper a correcting factor has had to be assumed in cases where the Carcel lamp is corrected for a difference of humidity.

Accuracy of Comparisons.—It is well to explain in giving the results of experiments that different limits of accuracy

* "Zeitschrift für Instrumentenkunde," vol. xv. 1895, p. 157.

† Séances of the French Physical Soc., May to July, 1884.

‡ "Rapport de Trois Lamps," Laporte & Jouaust, Bull. Soc. Inst. des Elect. 2nd série, tome vi. no. 58.

must be attributed to photometric measurements of different types of standards.

It is usual in giving photometric results to write down the fourth figure, but even in the *most favourable* circumstances this must be written small and the value considered liable to an error of + or - 0.1 per cent. In the case of the best comparisons of electric sub-standards this inaccuracy should not be occasioned by imperfection in the bench or photometer head nor yet to the electrical measurements, but must be attributed, in the author's opinion, chiefly to want of constancy in the individual who is making the photometric observations.

In some of the comparisons which are tabulated the electrical measurements are probably not so accurate as in others. The fuller appreciation, however, of the exact values of the national and international electrical units which has recently resulted from the labours of the International Conference, makes it possible now to attain an accuracy which leaves nothing to be desired from this point of view.

As matters stand now, undoubtedly the photometric comparisons in which the highest precision is attainable are those between properly seasoned electric glow-lamps of the same coloured light. With a potentiometer which is accurate to one part in 10,000 and a substitution method of photometric comparison * on a bench which can be read to 0.5 mm., an accuracy is attainable with a set of good sub-standards in which the fourth figure is almost definite.

When, on the other hand, comparisons are made against or between flame standards, the probable inaccuracy is greater.

How much the inaccuracy is must depend largely on the flame adjustments and the consistent behaviour of the standard in question. It also depends upon the accuracy of measurement of atmospheric conditions and the precise knowledge which we have of their effect on the light of the standard lamps.

It follows from this that a relatively large number of observations must be made, when using a flame standard,

* See "Photometry of Electric Lamps," by Dr. J. A. Fleming, M.A., F.R.S., Journ. Inst. Elect. Eng. vol. xxxii. p. 144.

if the same order of accuracy is to be attained that is possible with a much smaller number of electric comparisons. It must further be remembered when considering the question of photometric measurements to two or three parts in a thousand that the estimation of the height of the flame in some lamps, or the exact reproduction of the standard conditions, may not be identical when carried out by different observers. Hence it is conceivable that owing to this cause the observations in one laboratory on some flame standard may differ consistently by a small amount from those in another on the same standard. This, however, is not the case when electric sub-standard comparisons are made if the electrical measuring apparatus is accurate. To a certain extent, therefore (in some cases more than others), a flame standard needs to be "interpreted" when its absolute value is desired to a high accuracy.

The value of electric sub-standards comparisons thus becomes apparent. If (as is generally the case) a Laboratory has sets of electric sub-standards which have been compared at intervals for years with the primary flame standard whose value they represent, an opportunity is given for realising the absolute value of this unit to an accuracy which could hardly be attained with certainty by others who might endeavour to reproduce it in a single set of observations, however carefully made. When these electric sub-standards are intercompared through the medium of a travelling set of lamps, there is no reason why we should not obtain accurate knowledge of the relative values of the different units as each is interpreted in the country where it is the recognized standard.

The ratios between the four units of light given in the Table are the results of measurements which have been made at the specified laboratories in the four countries concerned. Other determinations were made previous to these *, but the standards used for obtaining the British Unit were of several different forms and the atmospheric conditions have not always been taken into consideration. I have deemed it

* For a discussion of these, see J. A. Fleming, "The Photometry of Electric Lamps," ref. cit.

desirable, therefore, to insert only the more recent determinations, in all of which the 10 Candle Harcourt Pentane Lamp has been used and atmospheric changes have been allowed for.

The Table is divided into two portions. Columns 1 to 9 give the various ratios obtained through the medium of electric lamps which have been measured at some or all of the laboratories. These have been chiefly initiated by the Americans, who have from time to time sent sets of lamps to Europe to have values assigned in London, Paris, and Berlin. It is not suggested that all the results given in the Table should receive equal weight. In some of the electric comparisons the conditions allowed of a greater accuracy than in others, when fewer lamps were employed and time only allowed a single set of measurements to be made.

Columns 10, 11, and 12 contain the values which resulted from the intercomparison of flame-standards undertaken at each of the laboratories. These were initiated by the International Commission on Photometry and gave a set of ratios which brought the knowledge of the relative values of the candle power units to within an accuracy of about \pm or -1 per cent. As in the case of the electric comparisons, the conditions in some cases probably allowed of a higher accuracy than in others—but the results of all the measurements have been tabulated in order that the bearing may be seen of each on the agreement which has been established.

The first series of ratios (columns 1 to 9) may therefore be regarded as representing the ratios of the standards as they are interpreted in the countries to which they belong, whilst in the second series we have the interpretation of the values of the standard lamps by experimenters who are not so accustomed to their manipulation.

Lines marked A, B, C, give the values of the standards in terms of the Pentane unit. D, E, F, give them in terms of the Hefner and similarly other sets are in terms of the Bougie Decimale and the Bureau of Standards Candle.

Without going into detailed comments upon the experiments from which each ratio in the table is derived it will suffice to say that as far as the electric lamp comparisons are

TABLE *giving determinations of the Ratios of*
Pentane = British. Bougie Decimale = French.

ELECTRIC						
COLUMN	1.	2.	3.	4.	5.	6.
Tests conducted by {	Sharp, 1903.	Paterson, 1905.	Fleming, 1905.	Hyde, 1906.	Laporte & Jouaust, 1907.	Laporte, 1907.
Number of Lamps.....	...	6	3	12	6	2
A <u>Hefner</u> Pentane Unit	0·89 ₀	0·88 ₅	0·89 ₃	0·90 ₈
B <u>Bougie Decimale</u> Pentane Unit	0·99 ₈	1·01 ₃	1·00 ₃
C <u>Bur. of Standards</u> Pentane Unit	1·01 ₆
D <u>Pentane Unit</u> Hefner	1·12 ₃	1·13 ₀	1·12 ₀	1·10 ₁
E <u>Bougie Decimale</u> Hefner	1·11 ₈	1·11 ₆	1·11 ₃
F <u>Bur. of Standards</u> Hefner	1·13 ₈
G <u>Pentane Unit</u> Bougie Decimale	1·00 ₂	0·98 ₇	0·99 ₇
H <u>Hefner</u> Bougie Decimale	0·89 ₄	0·89 ₆	0·89 ₈
I <u>Bur. of Standards</u> Bougie Decimale	1·01 ₈
J <u>Pentane Unit</u> Bur. of Standards	0·98 ₄
K <u>Hefner</u> Bur. of Standards	0·87 ₉
L <u>Bougie Decimale</u> Bur. of Standards	0·98 ₂

* Pentane values corrected to a humidity of

National Candle Power Units from 1903 to 1908.

Hefner=German. Bureau of Standards=U.S.A.

COMPARISONS.			DIRECT COMPARISONS.			
7.	8.	9.	10.	11.	12.	13.
Rosa, 1908. Spring.	Rosa, 1908. Autumn.	Laporte, 1908.	Paterson.	Liebenthal.	Perot & Laporte.	Photometric Commission, Zurich,* 1907.
12	6	11				
0·89 ₆	0·90 ₂	0·90 ₄	0·92 ₁	0·90 ₂
1·00 ₈	1·00 ₈	1·00 ₉	1·00 ₉	1·01 ₉	1·00 ₈
1·01 ₅	1·01 ₆
1·11 ₆	1·10 ₉	1·10 ₆	1·08 ₆	1·10 ₉
1·12 ₄	1·11 ₈	1·11 ₆	1·11 ₇	1·11 ₈
1·13 ₁
0·99 ₂	0·99 ₂	0·99 ₁	0·99 ₁	0·98 ₁	0·99 ₂
0·89 ₀	0·89 ₄	0·89 ₅	0·89 ₆	0·89 ₄
1·00 ₆
0·98 ₅	0·98 ₄
0·88 ₄
0·99 ₄

8 litres of water vapour per cubic metre of air.

concerned, greatest stress should be laid on the results in columns 4 and 7. This is partly on account of the large number of lamps employed, and also because of the more prolonged measurements made. Line J illustrates the high accuracy it is possible to secure in comparisons of this nature.

In the certification of glow-lamps in terms of the Hefner Unit the Reichsanstalt only give candle-power values to the nearest one per cent. If the average of 10 or 12 lamps is taken the error thus introduced is probably not great, but when the number is small, appreciable inaccuracies may be introduced into the mean, and the rather low value obtained by Fleming in 1905 may be attributed to the fact that only 3 lamps were tested *.

As regards the flame-lamp comparisons, it will be noticed that Perot and Laporte (Column 12) found a value for their Pentane lamp which was appreciably lower than that obtained by other observers. Except for this difference the agreement between the ratios is fairly close. The exceptionally close agreement shown in columns 10, 11, and 12 for the ratio Bougie Dec./Hefner can only be attributed to a coincidence, since in the experiments from which two of the three ratios were determined the observations varied between 3 and 4 per cent. from the mean.

The chief point of interest in these comparisons is the near coincidence of the value of the Bougie Decimale with that of the Pentane unit as indicated by lines B and G. An inspection of these values indicates that the Bougie Decimale, as interpreted by the Laboratoire Central, may be slightly larger than the Pentane unit—but the amount is less than 1 per cent. When we remember that, at present, the value of the Bougie Decimale depends on the interpretation of the Carcel lamp and the ratio between it and the platinum unit, determined by Violle in 1884, it must be admitted that this small apparent difference is well within the limits of the errors of experiment.

The second point to notice is the difference of 1·6 per cent.

* See Journ. Inst. Elect. Eng. vol. xxxviii. p. 311: Discussion by Dr. Fleming of the Author's paper on "Investigation of Light Standards etc."

between the units of the Bureau of Standards and the National Physical Laboratory. It is generally recognized* that the unit at present adopted by the Gas interests in the States is about 4 per cent. smaller than the Bureau of Standards unit. It will then be seen that by lowering the value of their unit by 1·6 per cent. the Bureau comes into exact agreement with this country and approximately halves the difference between the units employed by the Gas and Electrical industries in the States.

A further point of interest and importance which results from the comparisons (see line A in the table) is that the Hefner unit is in the ratio 9/10 to the new candle. The French authorities have for some time taken the ratio Hefner/Bougie Dec. as 0·895. It is of interest, therefore, to see from lines A and H how nearly the value for the Hefner unit in terms of both Pentane and Bougie Dec. units approaches the figure 0·90. The mean of all the ratios Hefner/Pentane comes to 0·90₀, and those of Hefner/Bougie Dec. to 0·89₅, so that although the comparisons between the Pentane and Bougie Decimale units indicate a difference of 0·8 per cent., the same units compared through the Hefner Standard only appear to differ by 0·5 per cent.

The author's acknowledgments are due to Dr. R. T. Glazebrook, F.R.S., Director of the National Physical Laboratory.

APPENDIX.

Copy of Announcement made in France, America, and Great Britain, relative to the proposed International Unit of Light.

In order to determine as accurately as possible the relations between the photometric units of America, France, Germany, and Great Britain, comparisons have been made at different times during the past few years at the Bureau of Standards, Washington; at the Laboratoire Central d'Electricite, Paris; at the Physikalisch-Technische Reichsanstalt, Berlin, and at the National Physical Laboratory, London.

* Report of Committee on Nomenclature and Standards, Annual Conv. Illum. Eng. Soc. Oct. 5, 1908, Dr. A. C. Humphreys.

The unit of light at the Bureau of Standards has been maintained through the medium of a series of incandescent electric lamps, the values of which were originally intended to be in agreement with the British unit, being made 100/88 times the Hefner unit.

The unit of light at the Laboratoire Central is the bougie decimale, which is the twentieth part of the standard defined by the International Conference on Units of 1884, and which is taken, in accordance with the experiments of Violle, as 0.104 of the Carcel lamp.

The unit of light at the Physikalisch-Technische Reichsanstalt is that given by the Hefner lamp burning at normal barometric pressure (76 cm.) in an atmosphere containing 8.8 litres of water-vapour per cubic metre.

The unit of light at the National Physical Laboratory is that given by the 10 candle power Harcourt Pentane lamp burning at normal barometric pressure (76 cm.) in an atmosphere containing 8 litres of water-vapour per cubic metre.

In addition to the direct intercomparison of flame standards carried out recently by the national laboratories in Europe, one comparison was made in 1906 and two in 1908 between the American and European units by means of carefully seasoned carbon filament electric standards, and as a result of all the comparisons, the following relationships are established between the above units.

The Pentane unit has the same value within the errors of experiment as the bougie decimale. It is 1.6 per cent. less than the standard candle of the United States of America, and 11 per cent. greater than the Hefner unit.

In order to come into agreement with Great Britain and France, the Bureau of Standards of America proposed to reduce its standard candle by 1.6 per cent. provided that France and Great Britain would unite with America in maintaining the common value constant, and with the approval of other countries would call it the International Candle. The National Physical Laboratory, London, and the Laboratoire Central d'Electricité, Paris, have agreed to adopt this proposal in respect to the photometric standardization which they undertake, and the date agreed upon for the

adoption of the common unit and the change of unit in America is April 1, 1909.

The following simple relations will therefore hold after that date :

$$\begin{aligned}\text{Proposed New Unit} &= 1 \text{ Pentane Candle.} \\ &= 1 \text{ Bougie Decimale.} \\ &= 1 \text{ American Candle.} \\ &= 1.11 \text{ Hefner Unit.} \\ &= 0.104 \text{ Carcel Unit.}\end{aligned}$$

Therefore 1 Hefner Unit = 0.90 of the proposed unit.

The Pentane and other photometric standards in use in America will hereafter be standardized by the Bureau of Standards in terms of the new unit. This, within the limits of experimental error, will bring the photometric units for both gas and electrical industries in America and Great Britain and for the electrical industry in France to a single value, and the Hefner unit will be in the simple ratio of 9/10 to this international unit.

The proposal to call the common unit of light to be maintained jointly by the national standardizing laboratories of America, France, and Great Britain, the "International Candle" has been submitted to the International Electro-technical Commission, and through it to all the countries of the world which are represented on that Commission.

It is to hoped that such general approval will be secured, and that in the near future the term "International Candle" for the new unit will have official international sanction.

DISCUSSION.

Dr. FLEMING said that it was interesting to hear that the chief Powers had come to an understanding with each other as to the Unit of Light. It must be remembered, however, that this proposed International Unit had no objective existence, and no greater value as a unit of comparison than the Hefner or Pentane units, to which it was related by an arbitrary definition. Dr. Fleming said that he greatly regretted that the National Physical Laboratory authorities had acquiesced in the adoption of a flame standard of light with all its difficulties and variabilities. Influenced as they are by atmospheric pressure, moisture, CO_2 , height of flame, composition of fuel, and number of persons in the photometric room, these flame standards could not possibly be considered as a final solution of the problem of obtaining a primary standard of light. What was really

required was the concrete realization of a permanent primary standard, which would be the standard of reference for secondary standards like the Fleming Ediswan Large-bulb glow-lamp standards, which he (Dr. Fleming) had introduced seven years ago. Dr. Fleming remarked that Mr. Paterson made only a very brief reference to M. Violle's work on the platinum standard and ignored altogether the careful work of Prof. Petavel carried out in 1899 in the Davy-Faraday Laboratory. Prof. Petavel's conclusions were that with suitable precautions the unit of illumination could be reproduced within 1 per cent. by means of the molten platinum standard. He asked Mr. Paterson if any attempt had been made at the National Physical Laboratory to repeat or extend Prof. Petavel's work, and if not why not? Investigations of this kind, which were difficult to carry beyond a certain point in private laboratories, were peculiarly the province of a State-aided institution like the National Physical Laboratory.

Dr. Fleming remarked that he was pleased to see that Mr. Paterson endorsed the conclusions which he (Dr. Fleming) had stated seven years previously in a paper read before the Institution of Electrical Engineers, viz., that properly prepared (large bulb) glow lamps constitute the best secondary standards. He (Dr. Fleming) had now employed for fourteen years secondary standards of this type, and had not found anything to surpass them in convenience and accuracy. The flame standards were unequally affected by changes in atmospheric pressure and moisture. Hence any figures for ratios such as are given in Mr. Paterson's paper are true only under certain accurately defined conditions of surrounding atmosphere which are very difficult to reproduce. Accordingly elaborate experiments to ascertain how many Hefners are equal to 1 Pentane is not a matter of nearly such importance as the construction of some final constant primary standard of light, and in his (Dr. Fleming's) opinion the most satisfactory form for this primary standard of light is to derive it from the light emitted normally by a defined area of some substance in a state of incandescence at a known fixed temperature. He was sure that many practical photometrists, especially those connected with the electric lighting industry, were not at all convinced that the best primary standard was a flame standard, or that the Pentane or Hefner units were a completely satisfactory solution of the problem of obtaining a Primary Standard of Light.

Dr. RUSSELL complimented the Author on his experimental results. The *bougie décimale* was the unit adopted by the International Congress of Electricians in 1889, and was defined to be the twentieth part of the Violle standard. He was not prepared to accept that it was equal to 1.11 Hefner Unit. Lummer's and Petavel's results rather discounted the importance to be attached to Violle's number. In connexion with Dr. Fleming's remarks he stated that the unit suggested by Waidner and Burgess had many advantages. They proposed to adopt as the unit of intensity the radiation from a square centimetre of a black body maintained at the temperature of the fusion of platinum. He referred also to the unit suggested by Steinmetz, and as Mr. Dyott was present he asked if he could give any information about this unit.

Mr. DYOTT said that his experiments had been made exclusively in connexion with Professor Steinmetz's magnetic arc. He had made no experiments on his photometric unit.

Mr. J. S. Dow remarked that the discussion had turned on the subject of standards rather than units. He wished, however, to make it clear that he did not condemn the Pentane standard as Dr. Fleming's quotation might, perhaps, suggest. Considered as a flame standard, he thought it ranked high, and gave very constant results when used in a scientific manner. Certainly flame standards had very serious drawbacks. But one must be cautious as yet in basing an incandescent standard upon results on a black body or other variety of radiation which might in the future be subject to modification. For instance, Féry had shown that some surfaces hitherto regarded as black exercised marked selective action; this seemed to suggest that the results of Lummer and others might need revision.

Whatever standard we adopted, the decision regarding the international unit was a very welcome one, and great credit was due to the Illuminating Engineering Society in the United States for taking the initiative. The co-operation between gas and electric authorities in both countries had rendered international action feasible, and proved that those connected with both illuminants could work together for the common good. The present step forward might appear to some to be but a small one, but it formed an important precedent. The ultimate goal was a single international unit, and it might be hoped that Germany would soon fall into line, for, as Mr. Gaster had recently pointed out, it was to her advantage, as a large exporter of glowlamps, to adopt the same unit as other countries. Meantime the adoption of the convenient round number 0.9 for the ratio between the Hefner and the proposed international candle was a very satisfactory compromise.

In this connexion we would like to enquire, however, whether Mr. Paterson felt sure that there was no physiological obstacle to comparing lights so different in colour as the Pentane and the Hefner with an exactitude of one per cent?

He had a recollection of some experiments carried out by M. Laporte in France which seemed, at that time, to suggest this possibility. It was found that when the ratio of the Hefner to the Pentane was obtained direct a value was obtained which differed consistently from that obtained by using the Carcel as an intermediate standard. This difference was then thought to be of physiological origin.

He understood, too, that Mr. Paterson had been experimenting with a series of incandescent lamps of graded efficiency (such lamps being used "in cascade" in order to avoid such an inconvenient colour difference as existed between an ordinary carbon filament and the Pentane lamp), but that the result of a "cascade" comparison was not always identical with a direct one. In view of his own experiences on this point (*Proc. Phys. Soc. London*, 1906, vol. xx.), it seemed conceivable that a small difference, physiological in origin, might be found to exist.

Dr. DRYSDALE thought that Mr. Paterson was to be congratulated on

his summary, and the international agreement arrived at was most welcome. As he understood the paper however, it was simply an attempt to obtain agreement between present existing units rather than standards, and left the matter of the best form of standard perfectly open. He thought that everyone having experience with flame standards would thoroughly agree with Prof. Fleming's condemnation of them, and there could be no doubt that the primary standard should be an incandescence one. He, however, did not agree with Dr. Fleming's suggestion of reviving the Violle standard. What was wanted in an incandescence standard was a definite area of a definite surface at a definite temperature. When the Violle standard was suggested we had little knowledge of the radiating properties of surfaces, or high temperature measurement, and therefore the only suitable thing was to take a very pure substance using its melting-point as a bench-mark of definite though unknown temperature. But everyone who had studied the history of the Violle standard was aware of the great difficulties of setting it up, and it had the disadvantage, according to Petavel, that the surface was dependent on the gas mixture used, besides an extremely short period of constancy and high expense. In the meantime we have realised that a perfectly black body is easily obtainable, and that it has perfectly definite radiating properties, we have the laws of Stefan and Wien, and optical pyrometry has advanced to a high degree of accuracy, and it therefore seemed decidedly preferable to suggest a unit area of a black body at a definite temperature. Mr. Jolley and he had come to the conclusion that a square cm. of a black body at a temperature of 2000° absolute would perhaps be a good unit and would be probably of the order of 100 candle-power. This temperature was probably pretty close to that of the ordinary carbon filament glow-lamp, so that there should be no colour difficulty and it should not be exceptionally difficult to maintain constant. If the temperature were measured by an optical pyrometer of say the Fery form based on the Stefan law, the deflection would be proportional to the 4th power of the absolute temperature, while the light according to Lummer and to integration from Wien's law was proportional to T^{12} . Hence the light would be proportional to the cube of the deflection only, and the probable error would not be large. Finally, a point in favour of the black body was the perfectly definite character of its spectrum, which made it a standard of colour as well as intensity and suitable for spectro-photometric comparison. As the surface would be that of a solid, it would be unnecessary to maintain it in a horizontal position, as with the Violle standard, and the amount of light could be easily varied by a diaphragm.

Dr. Drysdale said that he thought Mr. Dow had slightly misunderstood the nature of Prof. Fery's results, and it would be unfortunate if this should militate against the idea of the black body as a standard. There was no difficulty in obtaining a perfectly black body either by an enclosure or reflector. What Prof. Fery's recent experiments had shown was not that Kurlbaum's black radiators were at fault, but that he had been in error in assuming the perfect absorption of platinum-black with which his receiving bolometer was coated. This had necessitated an

increase of the constant in the Stefan formula from Kurlbaum's value of 5.32 to 6.32, but this was a point which could easily be settled and did not indicate any real difficulty in the use of the black body or the determination of its temperature, which could be simply extrapolated from known temperatures by the aid of the Stefan law.

Prof. C. H. LEES said that Prof. Petavel's recent work on the radiations from heated platinum strips suggested that he was not altogether satisfied with the Violle standard.

Mr. PATERSON expressed his interest in Dr. Fleming's remarks, but was not sure that in his criticism he appreciated the object of the paper. This was not to discuss the general question of light standards, but rather to deal with light units as they are at present. The unit which is officially or legally recognized in this country has been in existence for many years, and the National Physical Laboratory had no power to establish a new unit. A more constant and reliable standard of light than a flame was a great desideratum, but its value would have to be adjusted to be in agreement with the existing legal unit. Dr. Fleming had dwelt at some length upon the various disturbing factors for which allowance had to be made in using the Pentane lamp, but given long enough, the Author could and had reproduced the value of a set of electric sub-standards in terms of the Pentane lamp to an accuracy of + or - 0.1 to 0.2 per cent. He felt that years of work would be necessary on such a standard as Violle's before an accuracy of that order could be obtained; and if the laboratory had started work of this kind initially, they might still be without a definite unit of light.

On the general question of light standards raised by Dr. Fleming, he was inclined to agree with other speakers that the Violle platinum standard was not an ideal one. A serious objection he saw, even though the standard could be easily reproduced, was the colour of the light emitted. Molten metal at 1700° C. would give a considerably redder light than the low efficiency carbon filament, and now that temperatures of light sources were becoming so much higher, it seemed to him that the standard should not be retrograde in the matter of colour.

He suggested, in comment on Dr. Drysdale's desire for a black body standard, that the same colour difficulty would come in unless the furnace could stand a temperature of 1900° C. or 2000° C. Considerable difficulty would be experienced in keeping a clear reducing atmosphere in the furnace, and the fixing of the temperature in order to secure constancy of illumination to 0.1 per cent. would demand extremely sensitive and accurate temperature measurement. The proposal certainly seemed more promising than in the case of any other incandescent standard, and the possibility of obtaining direct horizontal radiation had very great advantages from the practical standpoint.

Dr. Russell criticised the ratio of 1.11 between the Bougie Decimale and the Hefner unit. The Author was not prepared to express an opinion on the point. By the Bougie Decimale in the paper the Author implied the unit as interpreted in France, which was the only country as yet which had nominally accepted it and professed to interpret it.

LIX. *On the Form of the Pulses constituting Full Radiation or White Light.* By ALBERT EAGLE, B.Sc., A.R.C.S., Imperial College of Science and Technology*.

ACCORDING to the modern theory of White Light founded by Gouy, and subsequently developed by Lord Rayleigh, Schuster, and others, White Light does not consist of periodic wave-trains of all wave-lengths, which are simply separated or "dispersed" when it is drawn out into its spectrum, but consists essentially of a succession of non-periodic pulses emitted independently of one another. It is out of such pulses that the spectroscope builds up the periodic wave-trains we observe in the spectrum.

If these pulses are all similar and follow one another quite at random, it follows that the distribution of energy in each pulse must be the same as the distribution of energy in the total succession of pulses. The distribution of energy in the spectrum obviously depends on the shape or form of the pulses making it. Lord Rayleigh has shown how† the distribution of energy in a pulse of any given form could be calculated, and calculates the distribution of energy for a pulse of the form $f(t) = e^{-c^2t^2}$. Other suggestions as to the form of the pulse have been made by other writers, and the distribution of energy which would be obtained from them has been calculated. In no case, however, has a form been hit upon which gave a distribution in accordance with fact.

The inverse problem—viz., from the distribution of energy, to find the form of pulse which would give rise to it—has not, as far as I know, been published, and its solution is the object of the present paper. We ought, however, to state that the problem is not one which admits of a definite solution, as the distribution of energy in the spectrum is independent of the relative phases of the infinitesimal harmonic components out of which the pulse may be considered to be built up; whereas its form must obviously

* Read June 11, 1909.

† Phil. Mag. vol. xxvii. p. 405 (1889), or Collected Works, vol. iii. p. 268.

depend as much on the relative phases of the components as upon their relative amplitudes.

Let $y = f(x)$ denote the form of the pulse in space. Lord Rayleigh has given * the now well-known relation

$$\int_{-\infty}^{\infty} f(x)^2 dx = \frac{1}{\pi} \int_0^{\infty} (A^2 + B^2) d\alpha, \quad \dots \quad (1)$$

where

$$A = \int_{-\infty}^{\infty} f(\mu) \cos \alpha \mu d\mu,$$

and

$$B = \int_{-\infty}^{\infty} f(\mu) \sin \alpha \mu d\mu.$$

The left-hand side of (1) is clearly proportional to the whole energy of the pulse, and the equation is to be interpreted as implying that the energy belonging to the waves comprised between α and $\alpha + d\alpha$ is proportional to $(A^2 + B^2) d\alpha$. The wave-length λ is of course $\frac{2\pi}{\alpha}$. Hence, if we are given that the energy between α and $\alpha + d\alpha$ is proportional to $F(\alpha) d\alpha$, we have

$$F(\alpha) = \left[\int_{-\infty}^{\infty} f(\mu) \cos \alpha \mu d\mu \right]^2 + \left[\int_{-\infty}^{\infty} f(\mu) \sin \alpha \mu d\mu \right]^2. \quad \dots \quad (2)$$

This equation solved for f will be the solution of the problem.

Two particular solutions may very easily be obtained, first when the pulse is an even function, and second when it is an odd function. In the first case (2) reduces to

$$\int_0^{\infty} f(\mu) \cos \alpha \mu d\mu = F(\alpha)^{\frac{1}{2}}, \quad \dots \quad (3)$$

and in the second case to

$$\int_0^{\infty} f(\mu) \sin \alpha \mu d\mu = F(\alpha)^{\frac{1}{2}}. \quad \dots \quad (4)$$

Now, in Fourier's Theorem

$$\phi(x) = \frac{1}{\pi} \int_0^{\infty} d\alpha \int_{-\infty}^{\infty} \phi(\lambda) \cos \alpha(\lambda - x) d\lambda,$$

* *Op. cit.*

let $\phi(x)$ be equal to zero if $x < 0$, and be equal to $f(x)$ if $x > 0$. Then we have

$$\frac{1}{\pi} \int_0^{\infty} d\alpha \int_0^{\infty} f(\lambda) \cos \alpha(\lambda - x) d\lambda \quad . \quad . \quad . \quad (5)$$

is equal to $f(x)$ for positive values of x and equal to zero for negative values of x . Hence

$$\frac{1}{\pi} \int_0^{\infty} d\alpha \int_0^{\infty} f(\lambda) \cos \alpha(\lambda + x) d\lambda \quad . \quad . \quad . \quad (6)$$

is equal to zero for positive values of x .

Adding and subtracting (5) and (6), we obtain

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \alpha x d\alpha \int_0^{\infty} f(\lambda) \cos \alpha \lambda d\lambda \quad . \quad . \quad (7)$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \alpha x d\alpha \int_0^{\infty} f(\lambda) \sin \alpha \lambda d\lambda \quad . \quad . \quad (8)$$

for positive values of x .

In much of what follows we shall frequently have an equation involving a cosine with a similar one involving a sine. In order to prevent having to duplicate such equations, we will write them with $\frac{\cos}{\sin}$, in which either the upper or the lower may be taken; but in one equation upper must be taken with upper and lower with lower.

Equations (7) and (8) may very conveniently be expressed in the form of the theorem:

$$\left. \begin{array}{l} \text{If} \quad \int_0^{\infty} f(x) \frac{\cos}{\sin} mx dx = \phi(m), \\ \text{then} \quad \int_0^{\infty} \phi(x) \frac{\cos}{\sin} mx dx = \frac{\pi}{2} f(m). \end{array} \right\} \quad . \quad . \quad . \quad (9)$$

In this form the equations are useful for finding certain definite integrals; for instance, from the result

$$\int_0^{\infty} e^{-ax} \cos mx dx = \frac{a}{m^2 + a^2},$$

we can at once deduce

$$\int_0^{\infty} \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-am}.$$

Further examples will not be given here, as the author hopes to discuss some results which may be obtained from equations (9) in another paper.

Equations (7) and (8) enable us to solve at once (3) and (4). Multiplying each side of (3) by $\cos \alpha x d\alpha$, and integrating from 0 to ∞ , we obtain by (7)

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F(\alpha)^{\frac{1}{2}} \cos \alpha x d\alpha, \quad . \quad . \quad . \quad (10)$$

which gives the form of the even pulse; similarly, the odd one is given by

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F(\alpha)^{\frac{1}{2}} \sin \alpha x d\alpha. \quad . \quad . \quad . \quad (11)$$

We will now determine the form of the pulses for some of the different formulæ that have been proposed for representing the distribution of energy in full radiation.

Both Lord Rayleigh's and Wien's formulæ are included under

$$E_{\lambda} d\lambda = C\theta^n \lambda^{n-5} e^{-\frac{c}{\lambda\theta}} d\lambda,$$

the former being obtained by putting $n=1$ and the latter by putting $n=0$. Transforming this so as to obtain the energy between α and $\alpha + d\alpha$ where $\alpha = \frac{2\pi}{\lambda}$, we get

$$F(\alpha) d\alpha = A\theta^n \alpha^{3-n} e^{-2b\alpha} d\alpha,$$

where $2b = \frac{c}{2\pi\theta}$.

Hence, dropping factors outside the integral sign, the forms of the pulses are given by

$$\begin{aligned} f(x) &= \int_0^{\infty} \alpha^{\frac{3-n}{2}} e^{-b\alpha} \frac{\cos}{\sin} \alpha x d\alpha \quad . \quad . \quad . \quad (12) \\ &= \frac{\Gamma\left(\frac{5-n}{2}\right) \frac{\cos}{\sin} \left\{ \frac{5-n}{2} \tan^{-1} \frac{x}{b} \right\}}{(b^2 + x^2)^{\frac{5-n}{4}}}. \end{aligned}$$

Putting $n=1$ for Lord Rayleigh's formula, the expressions reduce to

$$\frac{b^2 - x^2}{(b^2 + x^2)^2} \quad \text{and} \quad \frac{2bx}{(b^2 + x^2)^2}.$$

Taking Planck's formula for the distribution of energy,

$$E_\lambda d\lambda = \frac{C\lambda^{-5}d\lambda}{e^{\frac{c}{\lambda\theta}} - 1},$$

this transforms into

$$F(\alpha) d\alpha = \frac{A\alpha^3 d\alpha}{e^{2b\alpha} - 1}, \quad \text{where } 2b = \frac{c}{2\pi\theta} \text{ as before.}$$

Extracting the square root of this expression by expanding the denominator in ascending powers of $e^{-b\alpha}$, substituting the result in (10) and (11), and integrating term by term, we get for the form of the pulses

$$\begin{aligned} f(x) = & \frac{\cos \left\{ \frac{5}{2} \tan^{-1} \frac{x}{b} \right\}}{\sin \left\{ \frac{5}{2} \tan^{-1} \frac{x}{b} \right\}} \frac{1}{(b^2 + x^2)^{\frac{5}{2}}} + \frac{1}{2} \frac{\cos \left\{ \frac{5}{2} \tan^{-1} \frac{x}{3b} \right\}}{\sin \left\{ \frac{5}{2} \tan^{-1} \frac{x}{3b} \right\}} \frac{1}{(3^2 b^2 + x^2)^{\frac{5}{2}}} \\ & + \frac{3}{8} \frac{\cos \left\{ \frac{5}{2} \tan^{-1} \frac{x}{5b} \right\}}{\sin \left\{ \frac{5}{2} \tan^{-1} \frac{x}{5b} \right\}} \frac{1}{(5^2 b^2 + x^2)^{\frac{5}{2}}} + \dots \quad (13) \end{aligned}$$

To the eye these pulses have much the same form as the simple ones obtained from Lord Rayleigh's formula.

It is interesting to observe that all the pulses we have found satisfy the condition $\int_{-\infty}^{\infty} f(x) dx = 0$. This is obvious for the odd pulses; for the even ones, we have only to show that $\int_0^{\infty} f(x) dx = 0$.

Now $f(x)$ consists of one or more terms of the form

$$\int_0^{\infty} \alpha^p e^{-q\alpha} \cos \alpha x d\alpha.$$

This integrated with respect to x gives

$$\int_0^{\infty} \alpha^{p-1} e^{-q\alpha} \sin \alpha x d\alpha = \frac{\Gamma(p) \sin \left\{ p \tan^{-1} \frac{x}{q} \right\}}{(q^2 + x^2)^{\frac{p}{2}}} \quad (14)$$

which vanishes when taken between 0 and ∞ if p be positive.

On the electron theory these pulses are due to the radiation from moving electrons, and the function giving the form of the pulse as a function of t is the same as the function giving the acceleration of the electron at time t . Hence, to find the displacement at any time, we must integrate the function giving the form of the pulse twice. By integrating the left-hand side of (14) twice with respect to x , we observe that we merely change p into $p-2$. Hence, to integrate the right-hand side twice we have only to change p into $p-2$. Applying this to equations (12) and (13), we get the motion of an electron which will give rise to these pulses.

For the pulses $\frac{b^2-x^2}{(b^2+x^2)^2}$ and $\frac{2bx}{(b^2+x^2)^2}$, obtained from Lord Rayleigh's formula, the motion of the electron is given by

$$y = A \log(a^2 + t^2) \quad \text{and} \quad y = B \tan^{-1} t/a,$$

where $a = \frac{b}{V}$.

Equations (10) and (11) show that the pulses may be regarded as resulting from a superposition of a series of sine curves of all wave-lengths, each with a suitable amplitude so as to give the required energy distribution. This being so, we may expect that we shall be able to obtain a more general form of pulse by assigning an arbitrary phase to the component sine curves, without thereby altering the distribution of energy in any manner. That is, we may expect

$$f(x) = \frac{2}{\pi} \int_0^\infty F(\alpha)^{\frac{1}{2}} \cos \{ \alpha x - \phi(\alpha) \} d\alpha. \quad \dots (15)$$

to be a more general solution of (2) than (10) or (11). This in fact is so, as may be verified by splitting up (2) into the two equations

$$F(\alpha) \cos \phi(\alpha) = \int_{-\infty}^{\infty} f(\mu) \cos \alpha \mu d\mu$$

and

$$F(\alpha)^{\frac{1}{2}} \sin \phi(\alpha) = \int_{-\infty}^{\infty} f(\mu) \sin \alpha \mu d\mu,$$

and trying to find a solution of these—in the same manner in which (3) and (4) were solved—which at the same time satisfies both of them. Equation (15) will be the result.

It is no longer, of course, believed that white light consists of a succession of pulses all of exactly the same form. It consists rather of a succession of pulses capable of being represented by an equation with one or more arbitrary parameters, the number of pulses emitted per second which have their parameters lying within definite limits being given by some law analogous to the Maxwellian distribution law. But the form we have obtained may probably be looked upon as some mean or average form of the pulses, and may perhaps be of some value in determining, to a first approximation at least, by what intermolecular forces the free electrons in a substance must be acted upon in order to give the observed distribution of energy in the spectrum. Although, as we have seen, the problem does not admit of a definite solution, yet it is not impossible that physically all the pulses in white light may be (say) even ones. For instance, if an electron moves in a straight line against an opposing force which is a function of the distance, and which is insensible at the beginning of the motion but becomes sufficiently powerful to bring it to rest and reverse its motion, the pulse produced will obviously be an even one, under which conditions the form of the pulse for a given distribution of energy is unique.

LX. *Note on Terrestrial Magnetism.*

By G. W. WALKER, M.A.*

IN several papers dealing with an explanation of terrestrial magnetism by extraneous magnetic force, I have been surprised to find it assumed as obvious that if the earth had a large magnetic permeability, the effect of a given extraneous

* Read June 11, 1909.

force would be largely magnified, and that thus small forces might produce effects as large as those actually observed*.

The following considerations show that this view is quite erroneous.

Suppose we have a sphere of radius a and magnetic permeability μ , and let an extraneous magnetic field be represented by a potential $F P_n \frac{r^n}{a^{n+1}}$, P_n being a zonal harmonic of order n . The effect of the sphere is represented at outside points by the additional potential

$$\frac{Fn(1-\mu)}{\mu n + n + 1} P_n \frac{a^{n+2}}{r^{n+1}}.$$

We thus find that the normal force at the surface is altered in the proportion $\frac{\mu(2n+1)}{\mu n + n + 1}$ and that the tangential component of force at the surface is altered in the proportion

$\frac{2n+1}{\mu n + n + 1}$. The most likely case in the application to terrestrial magnetism is $n = 1$, and so for a high value of μ the normal force is increased in proportion 3 to 1, while the tangential component becomes nil.

For higher values of n the normal force is increased in a less proportion than 3 : 1. Only in the case $n = 0$ is the normal force increased in proportion $\mu : 1$. But an extraneous field which should be entirely radial in the vicinity of the earth does not appear to me possible.

It seems to me therefore desirable to point out that the assumption of large magnetic permeability of the earth is no real help in the explanation of terrestrial magnetic effects by extraneous magnetic forces.

* Cf. Arrhenius, *Kosm. Phys.* p. 984; Pflüger, *Phys. Zeit.* 1905, p. 415; and Humphreys, 'Terrestrial Magnetism,' Dec. 1908, p. 151.

LXI. *A Method of producing an intense Cadmium Spectrum, with a proposal for the use of Mercury and Cadmium as Standards in Refractometry.* By T. MARTIN LOWRY, D.Sc., F.C.S.*

OF the different line spectra that are available for spectroscopic standards—hydrogen, mercury, cadmium, &c.—the simplest and purest is undoubtedly the cadmium spectrum. The visible spectrum is made up of four strong lines (red, green, blue, and dark blue), which are so narrow and of such a high degree of purity in respect of the absence of satellites that they have been used by Michelson to produce interference-bands of an order of retardation that has apparently never been reached in the case of any other lines. Michelson's measurements of the wave-lengths of the three chief Cadmium lines :—

Cd red.....	6438.4722	10 ⁻¹⁰ metre.
Cd green.....	5085.8240	„ „
Cd blue	4799.9107	„ „

have indeed formed the standards from which all other wave-lengths have been deduced. It is therefore evident that the cadmium spectrum is destined to play an extremely important part in optical determinations of all kinds. Unfortunately, the difficulty of producing a cadmium lamp which shall burn steadily and give out light of high intensity has been so great that the four cadmium lines have been used only very occasionally in optical experiments.

Sodium.

The standard monochromatic light employed almost universally for refractometric and polarimetric measurements has been the yellow flame-spectrum of sodium, which has the advantage of being produced with very great readiness, but with all the drawbacks inseparable from the use of a doublet, instead of a single line, as a standard. Thus in determining the refractive index n_D of a liquid, the Pulfrich refractometer

* Read June 25, 1909.

gives readings for the less refrangible constituent, whilst a hollow prism mounted on a spectroscope gives an average value for the two constituents, unless indeed the resolution be sufficient to read them separately. In polarimetric work the double character of the sodium line renders it impossible to secure a proper extinction for large values of α_D , since one wave-length is transmitted with considerable intensity when the other is extinguished, and in addition there is always some uncertainty as to the "optical mass-centre" of the doublet, which may indeed vary in different types of sodium-lamp, on account of changes in the relative intensity of the two constituents*. It should also be noted that the sodium flame emits a considerable amount of light of other colours, which in accurate work, or in reading large rotations, must be removed by filtering through a coloured screen, or, better, by means of a spectroscopic eyepiece (Perkin).

Hydrogen.

The hydrogen lines,

$$H_\alpha \text{ (red), w.-l.} = 6560\cdot04,$$

$$H_\beta \text{ (blue), w.-l.} = 4861\cdot49,$$

$$H_\gamma \text{ (violet), w.-l.} = 4340\cdot66,$$

have been employed universally with the sodium doublet in refractometric work when dispersion-values were required. The choice has been wholly one of convenience and has no other merit to recommend it. The vacuum-tube, though easily fitted up, can hardly be considered seriously as a source of light. The red line is by far the strongest, and has been used with advantage to produce interference-fringes in measurements of length †, but would be utterly useless for polarimetric work in which the source of light must be viewed through a Nicol's prism set within 2° or 3° of the extinction position. The violet line is unpleasantly weak even for refractometric measurements, and demands the use of the

* Compare Landolt, *Optische Drehungsvermögen*, 1898, pp. 362 *et seq.*

† See, for instance, Tutton's measurements of the coefficients of expansion and of elasticity of crystal-plates (Phil. Trans. 1903, A. 202 p. 143). Compare also Tutton, Proc. R. S., June 10, 1909.

full power of a six-inch coil, with an efficient optical condenser, before readings can be made with any degree of comfort. The hydrogen spectrum has the further disadvantage of showing, at least in an ordinary vacuum-tube, an almost continuous back-ground of weak lines. Although, therefore, refractometers are regularly sent out with tables for the sodium and the three hydrogen lines—and no data whatever for light of any other wave-length—it is evident that this position is radically unsound and cannot be maintained permanently.

Choice of Standards.

The essential properties for a standard source of light are, (1) that it should be of sufficient intensity to be used for *all* the various types of optical measurements, so that, for instance, refractive indices and optical and magnetic rotatory powers may be determined for the same wave-lengths, (2) that it should be strictly monochromatic and as far as possible free from satellites, and (3) that it should be produced with sufficient readiness to render it generally available. These requirements, as has been shown, are only partially fulfilled by sodium light and fail completely in the case of the hydrogen spectrum. The purpose of the present communication is to suggest that the spectra of mercury and of cadmium fulfil most of the essential conditions outlined above, and to describe a method by which the cadmium spectrum may be rendered more generally available for spectroscopic and other optical work.

The suggestion—which is made on the basis of practical experience in the actual measurement of optical and magnetic rotations and of refractive indices for a large range of wave-lengths (see for instance Proc. Roy. Soc. 1908, 81. p. 472)—that sodium should give place to mercury and cadmium as a chief standard source of light, is fully supported by the theoretical considerations recently advanced by Bates ("Spectrum Lines as Light Sources in Polariscopic Measurements," Bureau of Standards, Bulletin, 1906, ii. p. 239) and by Nutting ("Polarimetric Sensibility and Accuracy," *ibid.* p. 249, "Purity and intensity of Monochromatic Light

Sources," *ibid.* p. 439). The former author has worked out a formula showing the errors due to the use of a doublet in polarimetry, and has redetermined the ratio of the sodium-yellow and mercury-green rotations for quartz; the latter has developed formulæ in reference to polarimetric sensibility, and spectral purity. The two points in these papers that bear directly on the practical problem now under consideration are, (1) the confirmation by Bates of the purity of the mercury green line, which gave very sharp readings in the case of a quartz plate of about $5\frac{1}{2}$ mm. thickness; this point is, however, seriously discounted by the fact that he professes to read the sodium doublet to $0\cdot0001^\circ$, and gives the ratio of sodium to mercury to six significant figures ($0\cdot850944:1$), (2) the statement by Nutting that on one basis of reckoning the "spectral purities" of cadmium green, mercury green, and sodium yellow, are represented by the ratios $\frac{1}{100,000}$, $\frac{1}{10,000}$, and $\frac{1}{700}$, whilst on another basis the "specific impurities" of the mercury green and sodium yellow lines are given by the ratios $\frac{1}{1,000,000}$, $\frac{1}{2000}$; these figures serve to show that the change of principal standard now proposed on the basis of practical polarimetric work is fully justified by minute spectroscopic tests on the lines themselves.

Mercury.

The use of the enclosed mercury arc as a source of light in spectroscopy dates back to 1860 (J. H. Gladstone, "On the Electric Light of Mercury," *Phil. Mag.* [4] xx. pp. 249-253), but its use in polarimetric measurements was apparently introduced by Disch (*Ann. Phys.* (4) xii. p. 1155) in 1903, who made use of the Arons lamp. The Bastian mercury lamp, which has been in use in my own laboratory since 1906, and at the Central Technical College since 1907, has the advantage of being a commercial article of much lower cost; it is constructed with a suitable resistance in the holder, so that it can be plugged into the ordinary lighting circuit without using a resistance-frame or any special leads. This lamp is unfortunately no longer on the market, though it is still constructed to order by the Brush Electrical Engineering Company; but

silica lamps of moderate price are promised which may prove to be as economical in working as, and even more efficient in illumination than, the earlier glass lamps.

Of the six chief mercury lines,

5790·49 } a yellow doublet,
5769·45 }

5460·97 a splendid green line,

4358·58 a strong violet line,

4078·03 } at the extreme limit of the visible spectrum,
4046·78 }

two, the green and the violet, have already proved to be of the utmost value in polarimetry, and are likely in the future to prove of equal value in the measurement of refraction and dispersion.

Their use in polarimetry has been due to the following considerations. For accurate measurements of the specific rotatory power of a substance, and to any even larger extent for tracing the course of chemical changes (isomeric change, sugar-hydrolysis, &c.) by polarimetric observations, it is essential to use an intense source of light in association with a very small half-shadow angle, since only thus can a maximum of sensitiveness be secured: it is also desirable to use a light of relatively short wave-length, in order that the actual readings may be large *, without incurring the loss of optical intensity and the fatigue which result from the use of blue light. The intense green mercury line, which can be read with a considerably smaller half-shadow angle, and gives readings about 15 per cent. larger than the sodium doublet, is therefore much superior to the traditional standard, apart altogether from the question of spectral purity. In the latter respect the contrast is extreme; in the case of quartz I have been able, without any noticeable loss of accuracy, to secure readings showing a total rotation of 50 right angles for the mercury green line, two independent series of determinations

* As a rule the specific rotation is doubled on passing from yellow to violet.

giving average values 4487.78° and 4487.79° ; sodium under similar conditions gave no extinction at all. The green mercury line promises, indeed, wholly to replace the sodium doublet as a chief standard in polarimetric work, and it is highly desirable that it should acquire as quickly as possible a like predominance in the measurement of refractive indices, and in all other optical determinations.

The violet mercury line has proved indispensable in the measurement of rotatory dispersion on account of its extraordinary brilliancy. In spite of the low sensitiveness of the eye for light of such small wave-length it has been found possible to read this line with a half-shadow angle of only 6° , and to secure series of readings (each an average of 10 settings) which only differed from one another by a hundredth of a degree. The violet line is less pure than the green, as it is accompanied by two satellites of smaller wave-length, but these are so weak that they cannot be seen at all in the polarimeter, and cannot, therefore, produce any large disturbance in the readings. The yellow doublet is made up of two lines separated by about three times as great an interval as in the case of sodium; for small rotations they may be read as one line, but I have also been able, by using a narrow slit, to read them separately; they are, however, altogether unsuited for general use.

For refractometer work the mercury lines are at least as easily available as those of hydrogen; a warmed vacuum-tube containing a drop of mercury gives the lines with greater brilliance than those of hydrogen, and it is therefore not unreasonable to suggest that—as a minimum concession to the correlation of optical measurements of various kinds—the use of $H\beta$ 4861 and $H\gamma$ 4341 shall be abandoned in favour of $H\delta$ 5461 and $H\epsilon$ 4359 in future refractometric work, and that tables for these wave-lengths shall be supplied as a matter of course with instruments of the Pulfrich pattern. It may be noted that the violet mercury and hydrogen lines differ by only 18 Ångström units, the mercury line having the longer wave-length: in a Pulfrich instrument the two lines are indistinguishable, but the edge that is read with a

hydrogen-mercury vacuum-tube (such as is sometimes sent out with the instrument) is due to mercury and not hydrogen. The adaptation of a polarimeter for use with mercury light costs about £2, with a further £3 for the lamp.

Cadmium.

The cadmium spectrum is much less easy to produce than that of mercury. Michelson made use of a strongly heated vacuum-tube with aluminium electrodes connected to platinum wires passing through the glass. This was improved upon by Hamy (*Comptes Rendus*, 1897, cxxiv. p. 749) who used a copper heating-jacket and external electrodes, thus avoiding the risk of cracking the hot glass by wires passing through it. I have had no personal experience of such lamps, but am doubtful whether they would give a sufficiently intense light for use in polarimetry. The amalgam lamp with an arc enclosed in silica can be made to give a splendid series of lines for use in spectroscopy, but I have found that it is useless for polarimetric work, since even the green cadmium-line can only be read with a half-shadow angle of nearly 20° . Apparently the current is carried mainly by the mercury, and the other metals show only weakly in the spectrum. An enclosed cadmium arc has been described by Stark & Küch (*Phys. Zeitschr.* 1905, vi. pp. 438-443), but does not appear to have come into general use.

The method set out below is not put forward as the ideal way of producing an intense cadmium spectrum, but rather as an intermediate stage in the development of the perfect cadmium lamp of the future. It was found that brilliant spectral lines could be sent into the polarimeter by using an arc burning between metallic poles rotating in opposite directions. Copper, for instance, gave a valuable series of lines, and brass electrodes were found to be very efficient for developing a zinc spectrum, the red line Zn 6364, and the three blue lines Zn 4811, Zn 4722, and Zn 4680, standing out very distinctly from the copper lines. A brilliant cadmium spectrum could be produced by melting the metal onto copper electrodes, but it soon burned off, and in any case it

was difficult to avoid a displacement of the readings by the appearance of the copper line Cu 5106, as a ghostly partner of the green cadmium line Cd 5086. After many unsuccessful attempts a workable method of producing the cadmium spectrum was found in the use of an alloy of silver and cadmium. It is perhaps not very widely known that these metals are isomorphous, and form an excellent series of alloys. These have the advantage that no eutectic is formed, the melting-points throughout the series lying above that of cadmium and over a considerable range approximating somewhat closely to the melting-point of silver. Thus whilst the addition of 28 per cent. of copper (mp. 1082°) lowers the melting-point of silver from 960° to 780°, the addition of 28 per cent. of cadmium (mp. 322°) only lowers the melting-point to 860°. Nearly 50 per cent. of cadmium must be added to lower the melting-point to 780°, and even a 60 per cent. alloy melts as high as 700°. These alloys, which can be turned up like pure silver, were supplied by Messrs. Johnson and Matthey in the form of rods $\frac{1}{4}$ inch in diameter and $1\frac{1}{4}$ inch long.

For spectroscopic work a tiny arc can be burnt quite steadily between the points of the rods, in great contrast to the behaviour of pure cadmium, which splutters very badly and gets choked up with oxide, even when the current is kept so small as not to melt the metal.

For polarimetric work a greater intensity of light is desirable, and this is obtained by using a heavier current and rotating the electrodes in opposite directions (compare Baly, *Spectroscopy*, p. 370) in order to maintain the arc in a central position. The rods of alloy were screwed for half their length into copper cylinders $\frac{5}{8}$ inch in diameter, which served the double purpose of cooling the electrodes—a point of some importance—and connecting them with the iron spindles by means of which the rotation was produced. When run at the highest intensity both rods become red hot, and one of the copper cylinders is usually luminous, but it is not desirable to over-run the arc since even if the electrodes do not melt the cadmium distils out irregularly and causes a certain amount of spluttering.

The electrodes are filed up before the arc is started, and are

carefully adjusted so as to run true to centre ; alternatively they may be allowed to burn until the ends are flat and then used without further attention except to adjust the length of the arc from time to time. The cadmium spectrum thus produced is of great brilliance—the green line is even brighter than that of mercury, and can be read with a half-shadow angle of 3° or less. From some points of view it would be a better chief standard than mercury green, as it is considerably brighter and gives readings about 15 per cent. higher, but in view of the greater trouble involved in producing the light, and persuading it to burn steadily, it is better to use it as a secondary line for the study of rotatory dispersion. The red and blue lines are also very bright and can be read quite easily. The dark blue line is of much less intensity, and is not likely to be widely used, as it is difficult to read, and does not differ sufficiently in wave-length from the light blue line to justify the extra trouble involved ; this observation applies, however, only to the existing arrangements, as it is quite possible that when a more powerful source of steady light is available the dark blue line may prove to be of considerable value in shortening the gap between Cd 4800 and Hg 4359.

Unlike copper the silver spectrum does not clash at all with that of cadmium ; the brilliant silver green lines

5471·72	} doublet (compare sodium)
5465·66	
5209·25	

are separated from the cadmium green, Cd 5086, by an interval nearly as great as that which separates the two cadmium blues, and the only other line that shows at all strongly in the spectrum is a line in the far-violet, perhaps Ag 4055.

In conclusion : It is suggested that the mercury line Hg 5461 should be used as chief standard in optical work of all kinds, and that dispersion should be measured from this line to Hg 4359 instead of from H_α 6561 or H_β 4861 to H_γ 4341. As secondary standards are suggested the flame spectra Li 6708 and Na 5893, purified spectroscopically,

together with the three cadmium lines Cd 6438, Cd 5086, Cd 4800, giving a well-distributed series of seven wave-lengths.

6708,	6438,	5893,	5461,	5086,	4800,	4359.
Red	red	yellow	green	green	blue	violet.

130 Horseferry Road, Westminster, S.W.

Note added July 1909.—I am glad to find that the desirability of a change in the standard wave-lengths for use in refractometry is apparently recognized by other workers: in particular, Dorn & Lohmann in their measurements of the refractive indices of liquid crystals (*Ann. Physik*, June 10th, 1909, [4] xxix. pp. 535-565) have used the series Li 6708, Na 5893, Hg 5461, Hg 4359, which is identical with a series that I am using for the measurement of the refractive dispersions of the alcohols and acids of the aliphatic series, and differs from the series of seven lines used in the measurement of rotatory dispersion only in the omission of the three cadmium lines.

DISCUSSION.

Mr. TWYMAN remarked that during the last few weeks he had seen a cadmium tube, similar to a mercury lamp, working with satisfactory results. Such tubes had been used for some time by Paschen & Röntgen. He agreed with the Author with regard to the greater use of mercury light for spectroscopic and similar purposes.

Dr. A. E. H. TUTTON said that he had been working lately with Cadmium tubes and found that they worked well for a considerable time. With regard to the measurement of refractive indices he was astonished that more use was not made in this country of the monochromatic illuminator, which he described some years ago. The instrument was used in Germany and gave satisfactory results.

The AUTHOR agreed with Dr. Tutton that the monochromatic illuminator had certain well-recognized advantages, but hoped that those who used it would make a point of setting it to the wave-lengths suggested in the paper. Uniformity in the measurement of optical properties could only be achieved by adopting some series of standard wave-lengths such as the one whose advantages he had indicated.

LXII. *On the Measurement of Wave Length for High Frequency Electrical Oscillations.* By ALBERT CAMPBELL, B.A.*

(From the National Physical Laboratory.)

[Plate XXXIII.]

§ 1. *Introduction.*

IN all work with high frequency electrical oscillations, such as for example in wave telegraphy, it is of the utmost importance to be able to determine with accuracy the wave-length actually employed, and for this purpose several types of wavemeter are now in common use. In order to ensure accuracy in such measurements, it was suggested some time ago by the Post Office Authorities that arrangements should be made for the calibration of wavemeters at the National Physical Laboratory. As the results of our first series of investigations in the matter appeared to be of general interest, we publish them here by the kind permission of Major O'Meara, C.M.G., Engineer-in-Chief to the Post Office. Our experiments comprised the construction and testing of a standard wavemeter, and the verification of an ordinary commercial wavemeter sent to us by the Post Office. I shall designate this instrument (A) and our standard wavemeter (B) respectively. While it is unnecessary here to go into the history of the subject, I should like to give one or two references to earlier work by other observers which gave us much assistance, namely the experiments of Pierce † and those of Gehrcke‡.

While our work was in progress an important paper by Diesselhorst § appeared, and the results there published were in ample confirmation of those we were obtaining, as also were the earlier results of Pierce.

* Read June 25, 1909.

† Phys. Review, vol. xxiv. p. 152, 1907.

‡ *Elektrotech. Zeitschr.* (26), p. 697, 1905

§ *Jahrbuch der drahtlosen Telegr. u. Teleph.* vol. i. (1908).

§ 2. *Description of Wavemeter (A).*

Wavemeter (A) is of the Dönitz type and consists of a variable air condenser, with a range from about 100 to 1070 mmfd. (micromicrofarads), a thermo-junction and galvanometer, and a series of coils (A 1, A 2, to A 10) of self inductances ranging from 0.76 to 2313 microhenries. The combination is capable of measuring oscillation frequencies over a range extending from about 20,000,000 down to 100,000 \sim per sec., or wave-lengths from $\lambda=15$ metres up to $\lambda=3000$ metres. It must be kept in mind, however, that the accuracy obtainable depends on the part of the condenser scale at which the reading is taken. For example, at a reading of 20 the frequency can barely be read to 0.5 per cent.

As the coils (A 1, A 2,) of (A) are of solid (not stranded) wire of diameters from 0.32 up to 3.2 mm., the values of their self inductances at the high frequencies are, for most of the coils, considerably lower than those obtained at ordinary frequencies of 0 to 1000 \sim per sec.

With a view to checking the results of the direct experiments by calculation from the measured inductance and capacity of a wavemeter in which the inductances would be much less affected by frequency, and thus to obtain a standard instrument for future use, we constructed a wavemeter (B) in which the coils are all of highly stranded wire.

§ 3. *Description of Standard Wavemeter (B).*

The general arrangement of (B) was similar to that of (A). The variable condenser (from 100 to 900 mmfd.) was of special design, with amberite washers to give very high insulation. It was kindly presented to the Laboratory by Dr. Alexander Muirhead, F.R.S. We added to it a direct reading scale, which can be read to 1 in 2000 at the upper end and to 1 in 200 at the lower. The scale was constructed by a very careful series of tests by Maxwell's Commutator Method and was found to be very uniform.

For the inductances, three coils (Q b, Q 1, and Q 2) were used. They were all wound on ebonite tubes with stranded wire (7/36^s), *i. e.* containing seven insulated strands, each of

diameter 0.19 mm. In each coil the terminals were brought to a considerable distance (18.5 cm.) from the centre of the coil by fixed leads run parallel to one another 1 cm. apart. In this way too close proximity between the coil and the condenser plates was avoided. A Duddell thermo-ammeter of 1.5 ohms resistance was used to complete the circuit of the coil and the condenser, and by observing its maximum deflexion the point of resonance was obtained. The sensitivity of course varies with the coil used and with the nature of the oscillatory circuit; it was found to be sufficient for the purpose in the experiments described below. Fig. 1 (Pl. XXXIII.) shows a photograph of the wavemeter and of one of the coils separately.

§ 4. *Measurement of the Self Inductances of the Coils.*

The self inductances of all the coils were measured at ordinary frequencies (0 to 1000 \sim per sec.) by a method specially designed for the accurate measurement of such low values (A. Campbell, Phil. Mag. Jan. 1908). The comparisons were made against a standard variable mutual inductance with ranges of 0 to 200 and 0 to 2000 microhenries, the lower range being readable to 0.01 microhenry. The absolute value of this was measured in terms of the new Laboratory Standard of mutual inductance, whose value has been calculated to a very high degree of accuracy (see Proc. Roy. Soc. A. vol. 79, 1907). The subdivision of the scale of the variable mutual inductance was effected by the help of Maxwell's method of comparing two mutual inductances, Its accuracy was verified by an independent method as follows. A mutual inductance coil was constructed with the primary and secondary circuits both of well stranded wire (10 and 20 strands respectively), and, with all the strands in each circuit in series, the value was adjusted to be equal to the 10 microhenry reading on the scale. Then, by taking p strands of the primary and q strands of the secondary, the value would be $\frac{pq}{200} \times 10$, and thus inductances of $\frac{1}{200}$, $\frac{2}{200}$, $\frac{3}{200}$, ... of the full scale reading were obtained. On testing these against the actual scale, the subdivision was found to be perfectly satisfactory.

The details of winding and values of the self inductances of the coils of Wavemeter (B) are given in Table I. In actual working these values had to be increased by the

TABLE I.

Coil.	Coil Diameter. cm.	Axial Length. cm.	Turns.	Self Inductance. Microhenries.
QB.....	13.5	3.6	40	360.3
Q1.....	7.4	3.9	48	170.7
Q2.....	7.6	1.7	28	57.8 ₂

addition of the measured self inductance of the rest of the circuit, consisting of the leads, ammeter, and condenser ; the total addition was about 0.5 microhenry, of which 0.1 microhenry was due to the condenser.

§ 5. *Tests of Standard Wavemeter (B) by Photographing Sparks.*

In order to obtain an absolute calibration of the standard wavemeter, the frequencies of the oscillations with which it was tested were determined by including a spark gap in the main oscillation circuit to which the wavemeter was loosely coupled and photographing the spark trains by help of a rotating mirror.

The apparatus was arranged as follows :—

The source of current was a small alternator whose frequency could be varied from 50 to 200 ~ per second ; this was connected through a small high voltage transformer to a spark gap with cadmium electrodes shunted by a glass plate condenser and a bare wire inductance in oil. A large variable air condenser consisting of six aluminium disks, each 100 cms. in diameter, could be put in parallel with the glass plate condenser, and, by suitable variation of the capacity and inductance in circuit, oscillations of frequencies from 300,000 to 1,200,000 ~ per sec. could be obtained. A

special camera (fig. 2) was constructed and mounted for rigidity on a long slab of sandstone. At one end of the camera was the rotating mirror, which was concave and of 200 cms. radius of curvature. It turned on a vertical axis and was driven by a small motor, the speed being kept constant by an arrangement described below. The other end of the camera had two branches which carried respectively the spark electrodes and a pair of guides allowing the plate carrier to be smoothly raised or lowered while the rotating mirror threw the images of the spark trains as streaks upon the plate. The distance from the mirror to the plate was 200 cms. The speed of rotation of the mirror was kept constant (usually at about 60 revs. per second) as follows. On the axis was mounted a commutator and this was connected with a condenser, a bridge, battery, and galvanometer, for Maxwell's Commutator method of measuring capacity. By applying a slight variable brake to the rotating axis, the galvanometer light-spot could be kept at zero. When this was the case the speed was steady, and could be measured with a counter or deduced from the value of the condenser and the resistance in the bridge. From measurements of a number of the spark train photographs on each plate, the average displacement from spark to spark was found; and the frequency was calculated from this displacement, the distance from the mirror to the plate, and the speed of rotation of the mirror.

While each photograph was being taken the reading of the standard wavemeter was also observed, care being taken to keep the coupling to the spark circuit very loose.

The value (n) of the frequency obtained from the spark photographs was in each case compared with the values (n_1) calculated from the measured values K and L of the capacity and inductance in the wavemeter circuit. Since the wave-meter coils have a certain amount of distributed capacity, it was necessary to take account of this. The required correction was made by Glazebrook and Lodge's formula (Cambridge Phil. Trans. p. 171, vol. xviii. 1899),

$$p^2 L K \doteq 1 - \frac{N-2}{N^2} \cdot \frac{k}{K},$$

where $p=2\pi n$, N =number of turns in coil, k =capacity from turn to turn, and the capacities and inductances are in absolute measure. The value of k was found by testing the capacities of coils with bifilar windings of wire similar to that in the coils used. The correction is small, being in no case more than 1 part in 1000. The results are given in Table II.

TABLE II.

Plates Nos.	n By Sparks ~ per sec.	n_1 From K & L of Standard Wavemeter. ~ per sec.
61 & 62	290,300	290,500
47	516,800	516,800
57 & 58	818,300	821,200
55	1,042,000	1,039,000

We may remark that the small differences between n and n_1 are quite within the limits of probable experimental error. On the same plate the value of n deduced from the various spark trains sometimes showed an extreme variation of 1·2 per cent. from the mean; in the best experiments the variation was about 0·5 per cent. from the mean. In general the mean of 5 to 10 spark trains was used and the average variation from the mean was from 0·2 to 0·6 per cent. in the value of n .

§ 6. *Comparison of Wavemeter (A) with Standard (B).*

The condenser of (A) was tested throughout its range by Maxwell's Commutator Method and gave the results shown in Table III, which shows the scale readings to be very nearly proportional to the capacity. At the higher readings the accuracy of measurement of K is here of the order of 2 or 3 parts in 1000.

TABLE III.

Reading. Degrees.	K mmfd.	K/Reading.
20	126	6.1
40	245	6.0 ₈
60	367	6.1 ₂
80	486	6.0 ₈
100	607	6.0 ₇
120	724	6.0 ₃
140	851	6.0 ₈
160	966	6.0 ₄
180	1075	5.9 ₃

The coils (A4, A5, to A9) were tested for self inductance (L_0) at ordinary frequencies. In Table IV. are given the values found (including the working circuit in each case). Approximate dimensions of the winding are also

TABLE IV.

Coil.	2a Coil Diameter. cm.	l Axial Length. cm.	d Wire Diameter. cm.	N. Turns.	Self inductance, microhenries.	
					L_0 .	L_∞ .
A4	5.2	10.7	0.32	32	21.6 ₂	19.5 ₃
A5	8.0	11.3	0.32	33	44.9 ₆	42.0
A6	5.1	5.6	0.09 ₁	55	100.5	97.5
A7	5.1	9.6	0.09 ₁	95	195.0	189.8
A8	7.6	5.3	0.05 ₈	90	531.8	524
A9	7.6	8.9	0.05 ₈	150	1039	1025

given. In the last column are given the values of L_∞ (*i. e.* the value of the self inductance for infinite frequency, assuming that the current in that case is practically confined to the skin of the wire). These values have been calculated

by the approximate formula (due to Heaviside):*

$$L_0 - L_\infty = \frac{26N^2ad}{1000l} \text{ in microhenries,}$$

where N = number of turns,

a = radius of coil,

l = axial length of coil,

d = diameter of wire,

all the dimensions being in centimetres.

The application of the formula to such short coils is not quite appropriate, however, as it has been deduced on the assumption that the solenoid is long.

The two wavemeters were then loosely coupled to the same oscillation circuit and simultaneous readings taken at various frequencies. From the calibration of (B) already established and the results in Table III. the effective values of the self inductances of the coils of (A) were deduced; they are given in Table V.

TABLE V.

Coil.	Comparison Frequency ~ per sec.	Effective L (deduced). Microhenries.	L. (Mean).
A4	1,125,000	20.1	19.9
	1,338,000	19.8	
A5	837,400	42.6	42.5 ₅
	975,500	42.5	
A6	665,100	99.1	99.1
A7	366,700	190.5	191.2
	394,800	190.5	
	466,800	191.1	
	473,400	192.0	
	665,100	191.8	
A8	284,500	530	53 ₁
	290,300	530	
	322,800	532.5	
A9	290,300	104 ₈	104 ₈

* Collected Papers, vol. i., p. 356.

From Table VI. it will be seen that the observed effective values of L (to be used in working the instrument) lie between L_0 and L_∞ in all cases but the last. The discrepancy here may be due to an error of 0.5 per cent. in determining n which might occur in consequence of the reduced sensitivity when using a coil of such high resistance as A 9.

The column headed L_n gives the values of L calculated from L_∞ for the actual values of n by means of Cohen's formula *. It will be noticed that this brings the observed and the calculated values of the inductances (at the high frequencies) considerably closer.

TABLE VI.

Coil.	Effective L .	L_n .	L_0 .	L_∞ .
A4	19.9	19.6	21.6 ₂	19.5 ₅
A5	42.5 ₅	42.1	44.9 ₆	42.0
A6	99.1	98.1	100.5	97.5
A7	191.2	190.8	195.0	189.8
A8	531	526	531.8	524
A9	104 ₈	1029	1039	1025

§ 7. Conclusion.

Thus it appears that within the limits of wave-length used, the wavemeter with coils of well stranded wire gave results in close agreement with theory, while in the case of the instrument with coils of solid wire the agreement was as close as could be expected, as the correcting formulas are only strictly applicable to long solenoids.

In conclusion I would express my best thanks for kind assistance to Major O'Meara and his staff; to Prof. R. Ll. Jones, Messrs. H. C. Booth and T. L. Eckersley, who skilfully aided in the experiments; and to Dr. Glazebrook for valued help and advice throughout the work.

* Bulletin, Bureau of Standards, vol. iv., no. 1, p. 177 (1907).

DISCUSSION.

Mr. W. DUDDELL said the paper was a valuable one because accurate experiments on self-induction and capacity at high frequencies were required. With reference to the photographs he asked if the Author had used his method to photograph arcs and, if so, with what results. It would be interesting to know how low it is possible to get the apparent resistance of a coil with high frequency currents by stranding the wire.

Dr. ERSKINE-MURRAY, referring to Mr. Duddell's remarks, said that in actual practice the resistance could not be reduced more than about 10 per cent. by stranding.

Mr. TAYLOR congratulated the Author, and pointed out that the wavemeters described could only be used at the transmitting station. Wavemeters were required which could be used at receiving stations. Referring to the question of stranding he pointed out that it was possible to overdo the stranding and obtain less satisfactory results than could be obtained by stranding with a fewer number of wires.

Mr. G. B. DYKE called attention to experiments similar to those described which were being carried out by Dr. Fleming at University College. With reference to the spark photographs he was surprised that an accuracy of 1 in 1000 could be obtained.

The AUTHOR in reply stated that as the whole distributed capacity of the inductance coils had very little effect on the frequency of resonance, the dielectric hysteresis of the ebonite would be negligible except as regards damping. Sparks appeared to give sharper and more accurately measurable photographs than vacuum tube discharges or arcs, except mercury arcs, which gave the clearest and best pictures.

LXIII. *An Electromagnetic Method of Studying the Theory of and Solving Algebraical Equations of any Degree.* By ALEXANDER RUSSELL, M.A., D.Sc., and J. N. ALTY, A.I.E.E., *Faraday House, London* *.

CONTENTS.

1. Introduction.
2. The Electromagnetic Method.
3. Quadratic Equations.
4. The Equation to Curves passing through the Neutral Points.
5. Cubic Equations.
6. Finding the Roots of an Equation.
7. Description of Apparatus.

1. *Introduction.*

THE electrical device recently invented by Mr. Arthur Wright enables us to find approximate values of the real

* Read June 25, 1909.

roots of an equation at once by simple mechanical and electrical operations. In order, however, to find approximate values of the imaginary roots it is necessary to perform certain analytical operations, and then apply the device to find the roots of an equation of a higher degree. The method described in this paper has the great merit of giving approximate values of all the imaginary roots as well as all the real roots. It is not capable of such high accuracy as the Arthur Wright device, and it cannot be directly applied when the indices of the powers of the unknown quantity are fractional. On the other hand, it is exceedingly instructive, as it shows how the numerical values of both the real and imaginary roots vary as the coefficient of any power of the unknown quantity in the equation is varied. The apparatus required is exceedingly simple, and is to be found in practically every physical laboratory. We have found it quite a suitable experiment to include in a laboratory course for first year students.

The method suggested itself to one of the authors when studying a series of papers by Mr. F. Lucas which are published in the *Comptes Rendus*, t. 106 (1888). The final electrical method (p. 1072) devised by Mr. Lucas is a practical one. A sheet of tinfoil is spread over a large flat plate of glass or other insulating material. If the equation is of the n th degree $n+2$ sources and sinks for electrical current are provided. These are arranged on a line at equal distances apart. The currents in $n+2$ wires touching at the sources and sinks are adjusted so that they have certain definite values. The method of calculating these values is practically the same as in the method described below. The equipotential lines are then traced out either by an electrochemical method or by the Kirchhoff and Carey Foster method. If the line of sources and sinks be taken as the axis of x , and the origin be chosen midway between the outer wires, the coordinates of the nodal points, that is, the points where an equipotential line intersects itself, enable us to write down all the roots, real or imaginary, of the given equation at once. If (x_1, y_1) be the coordinates of a nodal point, we can see from the symmetry of the arrangement that $(x_1, -y_1)$ will be the coordinates of another nodal

point. It follows from the method of adjusting the currents that $x_1 \pm y_1 \sqrt{-1}$ gives a pair of conjugate roots of the given equation. The real roots, therefore, are all given by the abscissæ of the nodal points lying on the axis of X. The method is not rigorous, as Mr. Lucas has not considered the magnitude of the error introduced owing to the finite size of the conducting sheet. It would be very laborious to apply in practice. In solving a biquadratic, for instance, the currents in five wires would have to be adjusted to given values before the equipotential lines could be mapped out.

We shall now describe an electromagnetic method, in which the horizontal field due to the earth's magnetism is used in an analogous manner to the conducting sheet in Mr. Lucas's method. A drawing-board with a slit cut in it, a few pieces of bell-wire, any form of "charm" compass, ordinary ammeters and rheostats or lamp-resistance boards such as are found in every physical laboratory can be utilized at once for the experiment (see § 7).

2. The Electromagnetic Method.

Let us suppose that we have to find the roots for the equation

$$f(x) \equiv a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0. \quad (1)$$

We first, by the ordinary methods given in books on algebra, resolve the expression

$$\frac{f(x)}{(x-b_1)(x-b_2) \dots (x-b_n)}$$

into partial fractions. The numbers b_1, b_2, \dots, b_n are any convenient numbers so chosen, however, that

$$b_1 + b_2 + \dots + b_n = -a_{n-1}/a_n. \quad (2)$$

For example, if the second term of $f(x)$ be missing we must choose b_1, b_2, \dots so that $\Sigma b = 0$.

We thus obtain

$$\frac{f(x)}{(x-b_1)(x-b_2) \dots (x-b_n)} = a_n + \frac{A_1}{x-b_1} + \frac{A_2}{x-b_2} + \dots + \frac{A_n}{x-b_n}, \quad (3)$$

where, by (2),

$$A_1 + A_2 + \dots + A_n = 0, \quad (4)$$

and

$$\left. \begin{aligned} A_1 &= \frac{f(b_1)}{(b_1-b_2)(b_1-b_3) \dots}, & A_2 &= \frac{f(b_2)}{(b_2-b_1)(b_2-b_3) \dots} \\ \text{and} & & A_n &= \frac{f(b_n)}{(b_n-b_1)(b_n-b_2) \dots (b_n-b_{n-1})} \end{aligned} \right\}, \quad (5)$$

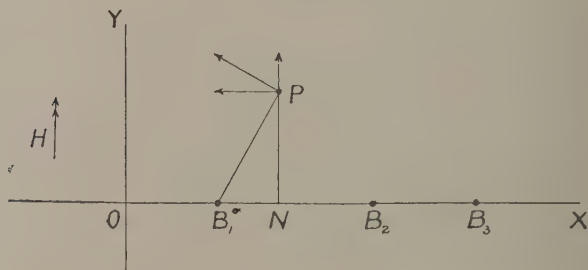
Let us now consider the magnetic field round a long vertical wire carrying a current of C amperes, and let us suppose that the earth's horizontal field in the neighbourhood is uniform, and that its horizontal intensity in c.g.s. units is H . The magnetic force at any point P at a perpendicular distance of r centimetres from the axis of the wire will be the resultant of a force $C/5r$ acting at right angles to the plane containing r and the axis of the wire and a force H directed to the magnetic pole. There is always a neutral point* on the line through the axis of the wire perpendicular to the magnetic meridian. If x be the distance of this point from the axis

$$x = C/(5H). \quad \dots \dots \dots (6)$$

This formula is utilized in a well-known rough laboratory method of measuring H when C is known or *vice versa*.

Let us now suppose that we have n vertical wires arranged in a plane perpendicular to the magnetic meridian, and let

Fig. 1.



them cut the plane of the paper perpendicularly at $B_1, B_2, \dots B_n$ (fig. 1) which are at distances $b_1, b_2, \dots b_n$ from O . If $C_1, C_2, \dots C_n$ be the values in amperes of the currents in the wires and H the horizontal intensity of the earth's

* A. Russell, 'The Electrician,' vol. xxxi. p. 282 (1893).

magnetic field, the components X and Y of the resultant magnetic force at $P(x_1, y_1)$ will be given by

$$-X = \frac{C_1}{5r_1} \cdot \frac{y_1}{r_1} + \frac{C_2}{5r_2} \cdot \frac{y_1}{r_2} + \frac{C_3}{5r_3} \cdot \frac{y_1}{r_3} + \dots$$

and

$$Y = H + \frac{C_1}{5r_1} \cdot \frac{x_1 - b_1}{r_1} + \frac{C_2}{5r_2} \cdot \frac{x_1 - b_2}{r_2} + \dots,$$

where $r_m^2 = (x_1 - b_m)^2 + y_1^2$.

Hence multiplying X by ι and subtracting we get

$$\begin{aligned} Y + X\iota &= H + \frac{C_1}{5r_1^2}(x_1 - b_1 - y_1\iota) + \frac{C_2}{5r_2^2}(x_1 - b_2 - y_1\iota) + \dots \\ &= H + \frac{C_1/5}{x_1 + y_1\iota - b_1} + \frac{C_2/5}{x_1 + y_1\iota - b_2} + \dots \end{aligned}$$

At a neutral point the resultant magnetic force is zero, and therefore both X and Y are zero. Hence, if x_1 and y_1 are the coordinates of a neutral point, $x_1 + y_1\iota$ is a root of the equation

$$0 = H + \frac{C_1/5}{x - b_1} + \frac{C_2/5}{x - b_2} + \dots \quad (7)$$

Comparing this with equation (3) we see that if we adjust the values of the currents so that

$$C_1 = 5H \cdot A_1/a_n, \quad C_2 = 5H \cdot A_2/a_n, \dots$$

$$C_n = 5H \cdot A_n/a_n,$$

then $x_1 + y_1\iota$ is a root of the equation $f(x) = 0$, and therefore $x_1 - y_1\iota$ is also a root.

It follows from (4) that $\Sigma C = 0$, and therefore only $n-1$ ammeters and only $n-1$ rheostats are required.

As an introduction to the method let us consider the theory of quadratic equations.

3. Quadratic Equations.]

Let us suppose that the equation is

$$x^2 + bx + c = 0.$$

In this case it is convenient to write

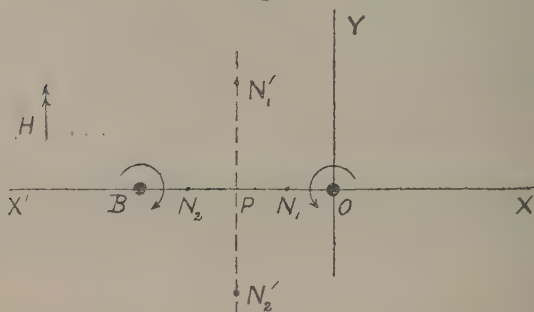
$$\frac{x^2 + bx + c}{x(x+b)} = 1 + \frac{c/b}{x} - \frac{c/b}{x+b},$$

and, therefore,

$$H \frac{x^2 + bx + c}{x(x+b)} = H + \frac{(5Hc/b)}{5x} - \frac{(5Hc/b)}{5(x+b)} \dots (8)$$

In fig. 2 let us suppose that the plane of the paper represents a horizontal plane, and that H is the direction of the earth's magnetic force. Let us also suppose that two long

Fig. 2.



vertical wires cut the plane of the paper perpendicularly at O and B respectively, the length of BO being b centimetres and the magnitude of the current C , in amperes, in the O wire being $5Hc/b$, and the B wire carrying the return current $-5Hc/b$. If the direction of the current C be into the paper at B and out of it at O , the circular lines of force due to the currents in each wire will act in the directions of the curved arrow-heads shown in the figure.

We shall now consider how the numerical values of the roots of the quadratic equation alter as c , and therefore also C , increases from zero to infinity, b which we suppose to be positive remaining constant.

We have already shown that the coordinates of the points at which the resultant magnetic force is zero, that is, the neutral points, determine completely the numerical values of both the real and imaginary roots of the equation.

When c is very small and positive the neutral points lie between B and O . For a particular value of c , for instance, they are at N_1 and N_2 on the axis of X . We see at once from symmetry that $ON_1 = N_2B$. If the length of ON_1 on

the same scale that BO is b be $-x_1$, the roots of the equation for this value of c are both real and equal to $-x_1$ and $-b+x_1$ respectively. As we increase the value of c , and therefore of the current in the wires, the neutral points N_1 and N_2 approach one another and coincide at P. In this case the roots are each equal to $-b/2$. For greater values of c the neutral points cannot possibly lie on BO as the resultant magnetic force due to the currents at all points of this line is greater than H. From symmetry the neutral points lie on the line N_1PN_2 bisecting OB at right angles. For a particular value of the current they are at the points N_1 and N_2 on this line. If $PN_1=y_1$ then $PN_2=-y_1$ and the corresponding roots of the equation are $-b/2+y_1\sqrt{-1}$ and $-b/2-y_1\sqrt{-1}$. The real part of the imaginary roots is therefore independent of c , but the coefficient of $\sqrt{-1}$ increases as c increases.

When c is negative the curved arrow-heads in fig. 2 must be drawn in the opposite directions. N_1 will now be at a distance x_1 from O along OX, and N_2 will be at a distance $-b-x_1$ from O along OX'. The roots in this case, therefore, are always real and of opposite sign, and continually increase numerically as c increases.

When b is negative the point B in fig. 2 will be to the right of OY, and the discussion of the roots in this case is equally simple.

4. *The Equation to Curves passing through the Neutral Points.*

From the preceding discussion it will be seen that the locus of the real roots of an equation determined in this way is the axis of x . Fig. 2 shows that for a quadratic equation when b is positive the locus of the points giving the imaginary roots is the straight line $N_1'N_2'$. Similarly when b is negative it is the parallel line given by equation $x=b/2$. It is important to know the equations to curves on which the neutral points must lie in the general case.

Let us suppose that $x+yi$ is a root of the equation $f(\xi)=0$. In this case we have

$$f(x+yi)=0,$$

and, therefore, by Taylor's theorem,

$$f(x) - \frac{y^2}{2} f''(x) + \frac{y^4}{4} f^{iv}(x) - \dots \\ + iy \left\{ f'(x) - \frac{y^2}{3} f'''(x) + \dots \right\} = 0.$$

This equation can be satisfied either by

$$f(x)=0, \quad \text{and} \quad y=0, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or by

$$f(x) - \frac{y^2}{2} f''(x) + \frac{y^4}{4} f^{iv}(x) - \dots = 0 \quad . \quad (10) \quad \left\{ \right.$$

and

$$f'(x) - \frac{y^2}{3} f'''(x) + \frac{y^4}{5} f^{v}(x) - \dots = 0 \quad . \quad (11) \quad \left\{ \right.$$

The equations (9) obviously give the real roots of the equation which must all lie on the line $y=0$, that is, on the axis of X . The points of intersection of the equations (10) and (11) determine the imaginary roots.

As equation (11) does not contain the constant term of the equation $f(x)=0$, it follows that it gives the curve locus of a series of imaginary roots of the equations formed by varying c . Where the curve represented by equation (11) cuts the axis of x we have $f'(x)=0$. At these points there are at least two equal roots.

For the quadratic equation $x^2+bx+c=0$, (11) becomes $2x+b=0$, and this is the equation to the line of neutral points $N_1'PN_2'$ shown in fig. 2.

5. Cubic Equation.

Let us suppose that the equation has been reduced to the form

$$x^3 - b^2x + c = 0. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

We easily find that

$$H \frac{x^3 - b^2x + c}{x(x^2 - b^2)} = H + \frac{C}{5(x-b)} - \frac{2C}{5x} + \frac{C}{5(x+b)}, \quad (13)$$

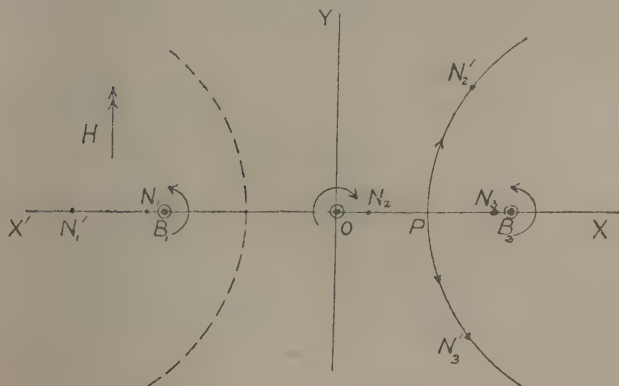
where $C=5Hc/2b^2$.

In this case (fig. 3) three long vertical wires are used. Let them cut the plane of the paper at right angles at B_1 , O ,

and B_3 respectively, and let $B_1O = OB_3 = b$. The currents in the wires passing through B_1 and B_3 must each be made equal to C , and the current in the wire passing through O should be the return current $2C$. The arrow-heads indicate that the direction of the current flow is out of the paper at B_1 and B_3 and into it at O . H denotes the direction of the earth's magnetic field.

Let us now suppose that c increases uniformly from zero to infinity. When c is small, and therefore the currents are small, two neutral points N_2 and N_3 obviously lie between O and B_3 and a neutral point N_1 lies along B_1X' .

Fig. 3.



As the current increases the neutral points N_2 and N_3 approach one another and the neutral point N_1 moves along B_1X' . For a particular value of c , N_2 and N_3 coincide at P , and we have two equal roots. For greater values of c the neutral points N_2 and N_3 move uniformly along the curve $N_2'PN_3'$, one being as much above the line XX' as the other is below it. But the neutral point N_1 always moves along the axis of X .

The equation (12), therefore, has always three roots. When c is small they are all real, two of them being positive and one negative. For a certain value of c the two positive roots become equal, and for greater values of c we have two conjugate imaginary roots and one real negative root. Both

the real and imaginary parts of the conjugate roots continually increase as c increases.

The equation to the curve $N_2'PN_3'$ in fig. 3 can be found at once from (11); it is

$$3x^2 - y^2 = b^2. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The negative branch of this hyperbola indicated by the dotted curve in fig. 3 gives the locus of the neutral points when c is negative. In this case there is obviously always one real positive root. Increasing the value of b in (12) is equivalent to putting the wires B_1 and B_3 further apart. This obviously increases the limits of the values of the positive real roots. When b is negligibly small we see by (14) that the locus of the neutral points is the two straight lines represented by

$$(y - x\sqrt{3})(y + x\sqrt{3}) = 0.$$

Hence the imaginary roots are of the form $x_1 \pm x_1\sqrt{3}\sqrt{-1}$ where $-2x_1$ is the value of the real negative root.

Similarly we can discuss the roots of the equation

$$x^3 + b^2x + c = 0.$$

In this case it will be found that the currents in the wires through B_1 and B_3 are unequal, and that the equation has always two imaginary roots, the neutral points lying on the hyperbola $y^2 - 3x^2 = b^2$, which is conjugate to the hyperbola shown in fig. 3.

Equations of the fourth and higher degrees can be discussed in like manner. To get the most instructive results care has to be taken to choose the distances between the wires so that the analytical expressions for the required currents may be as simple as possible. If this be not done analytical difficulties will often be encountered in interpreting the results.

6. *Finding the Roots of an Equation.*

The great and so far as we know the unique advantage of this method is that it enables us to find the imaginary as well as the real roots of an equation almost at once. In equations occurring in many physical problems it is the latter roots which we desire to find, and this method enables

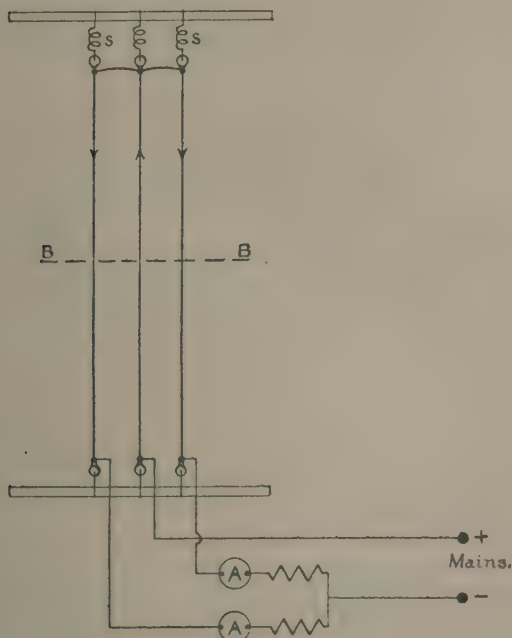
the physicist to find quickly approximate values of these roots.

If due precautions are taken the maximum inaccuracy of this method need not exceed one per cent. This is the accuracy obtainable by careful students, who need have no previous experimental training, in finding H by measuring the distance of the neutral point from a long vertical wire carrying a known current.

7. *Description of Apparatus:*

We shall now describe the simple apparatus we use for teaching purposes. In fig. 4 the arrangement of the apparatus

Fig. 4.



for solving a cubic equation is shown. S, S represent springs attached to insulators. These are necessary in order to keep the wires taut. The wires are No. 16 single cotton-covered

wire, and pass through a long slit cut in a drawing-board BB. The ammeters A are any of the ordinary ammeters used in the laboratory reading up to 10 amperes, and are in series with lamp-boards and rheostats. The currents are adjusted to the required values, and a sheet of sectional paper with a slit in it is put on the board. By means of an ordinary charin-compass the neutral points can then be readily found. It is advisable to make a little sketch of the lines of force near the neutral points, as this is a help in indicating their true position. The coordinates of these points read off from the sectional paper give the real and imaginary roots of the equation.

LXIV. *An Instrument for Measuring the Strength of an intense horizontal confined Magnetic Field.* By F. W. JORDAN, B.Sc.*

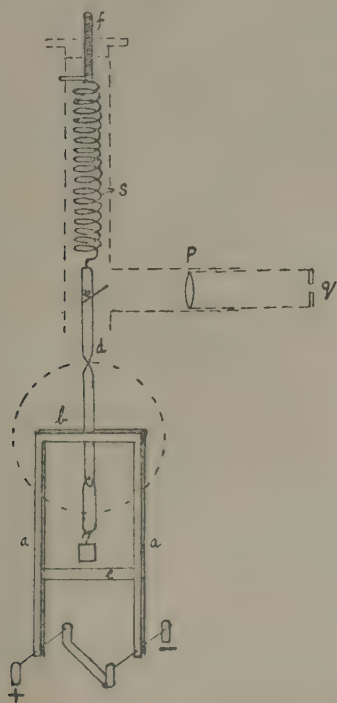
THE field was produced in the gap between the conical pole pieces of a large electromagnet resembling a split toroid. The experiment consisted in measuring directly the transverse force on a conductor, traversed by a current in a direction at right angles to the field. The close agreement between the results obtained by this method and those by the ballistic method, shows that the following apparatus may be relied upon to measure the strength of intense confined fields to at least 1 part in 200.

Two copper strips of uniform width, each cut to form three sides of a rectangle $ab a$, were fastened together and connected by short tinsel leads to terminals, so that the current could be sent in the same direction through each of the insulated conductors. Bending of the long strips aa was prevented by a thin connecting plate of mica e . The lower part of a vertical strip d was fastened to the centre of the horizontal conductors b and the upper end was suspended from a helical spring s of phosphor bronze wire. The tension of this spring was adjusted by a screw f to raise the con-

* Communicated by S. Skinner, M.A. Read June 25, 1909.

ductors to the sighted position, where a small circular hole in the vertical strip appeared to be bisected by a near horizontal wire. In the latter adjustment, parallax was avoided by using a single lens microscope p with a horizontal slit q for the eye aperture. The vertical motion of the conductors was limited by a projection moving between stops.

Fig. 1.



To make a measurement the apparatus was arranged so that the horizontal conductors were at right angles to the field at the centre of the gap. A known copper weight was suspended from the centre of the horizontal conductors and the strength of the current was adjusted to raise these to the

sighted position. At the same time the apparatus was moved, if necessary, to a position where the transverse forces on the vertical conductors were equal and opposite.

The strength of the field

$$F = \frac{mg}{lC} \text{ gausscs}$$

where mg dynes is the force on the horizontal conductors b .

l cm. is the sum of the distances between the centres of the vertical conductors a .

C is the current in absolute units.

The helical spring showed an extension of 1.2 cm. per gm. and a load of 0.005 gm. produced an observable extension. The weights ranged from 1 gm. for fields of about 1000 gausscs to 8 gm. for fields of about 10,000 gausscs. In a gap 1 cm. wide the conductors became slightly unstable and drifted slowly to the near face of the gap. This initial movement was checked by stretching two hairs on either side of the vertical strip.

The distance l as measured with a travelling microscope was found to be constant to 1 part in 1000 over the upper part of the conductors.

The current was measured with an ammeter to 1 part in 500 and ranged from 0.7 to 1.5 ampères.

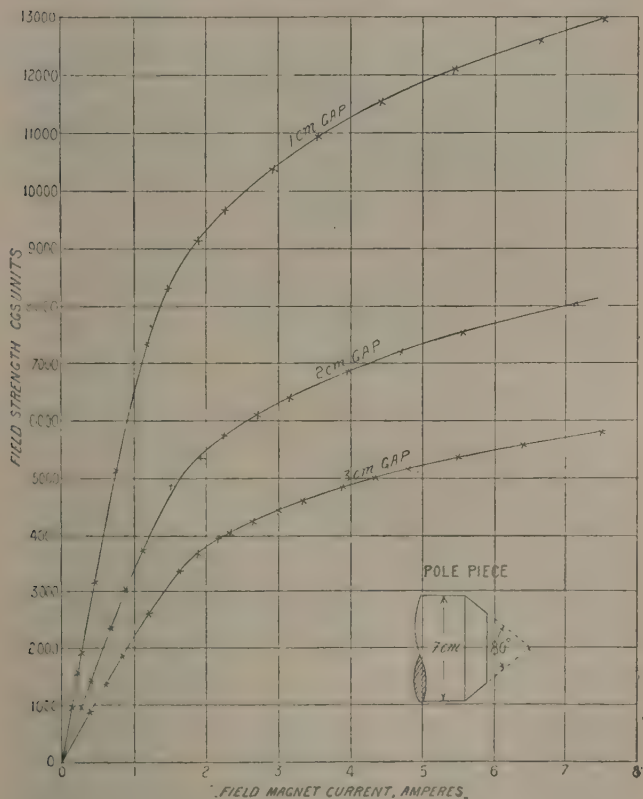
The tinsel leads were situated at right angles to the plane of the conductors and their terminal attachments were separated by the same mean distance as the centres of the vertical conductors. The maximum field strength in the direction of the leads was about 6 per cent of the field strength at the centre of the 3 cm. gap and 2 per cent at the centre of the 2 cm. gap. Thus the forces on the leads in this particular experiment were small and quite negligible.

The strength of the field in the gap itself was uniform to about 2 per cent., and since the diameter of the test coil in the ballistic method was made equal to the length of the horizontal conductor, it follows that no appreciable difference between the results of the two methods was to be expected.

The minimum field strength measured was about 1000

gausses and the maximum field strength attained in the 1 cm. gap was about 13,000 gausses. The results obtained by this apparatus agreed with those of the ballistic method

Fig. 2.



to about 1 part in 400. The field strength curves for gaps of three different widths were constructed from the results and are shown as an example of use of this instrument.

South Western Polytechnic Institute, Chelsea.

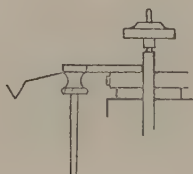
26 May, 1909.

LXV. *On a Method of Determining the Sensibility of a Balance.*

By J. H. POYNTING, *Sc.D., F.R.S.*, and G. W. TODD,
*M.Sc.**

IN the method, as we have arranged it, a small frame (fig. 1, end view) is fixed at the centre of the beam of a 16-inch Oertling balance. This carries two Vs about 2 cm. apart, and in the Vs lies a straight wire or fibre about $3\frac{1}{2}$ cm. long, parallel to the beam and level with the central knife-edge. This wire takes the place of the ordinary rider, and

Fig. 1.



End view of V frame fixed to balance-beam.

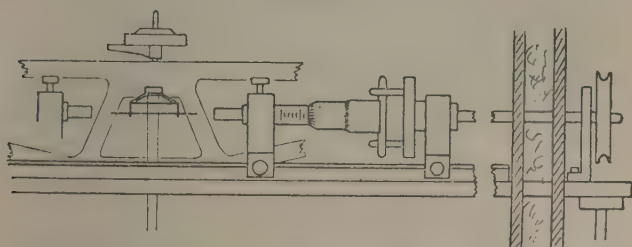
we shall call it "the rider." Its weight is determined before use as accurately as possible by weighing on an assay balance. The sensibility is determined by moving the rider either to right or left a measured distance. If this distance is d , if the half length of beam is b , and if the weight of the rider is R , the movement is equivalent to an addition of weight to one pan, Rd/b .

In order to move the rider a definite distance a stout horizontal rod (fig. 2) passes through the balance-case from side to side without contact with the case, and is supported at its ends outside, and independent of, the case. It is parallel to the beam and a little lower than the V frame. On the rod are fixed horizontally two Brown & Sharp micrometer-screws divided to $\cdot 01$ mm. and allowing an estimate of $\cdot 001$ mm. Their axes are in one line coinciding with the axis of the rider, and they are fixed so that one can bear against one end and the other against the other end of the rider. Their ends

* Read June 25, 1909.

are plane and the ends of the rider are bluntly pointed. Each micrometer screw-head has a cross piece fixed on it, and a fork which can be rotated about an axis in the continuation of the axis of the screw by a pulley outside the case, can engage with the cross piece, and so advance or withdraw the screw. The pulley is worked by an endless string passing to a pulley at the side of the observer, who is about 2 metres in front of the balance. The micrometer divisions are illuminated and each micrometer is viewed by its own telescope. The position of the balance-beam is read

Fig. 2.



Arrangement of right-hand micrometer-screw.

by a double-suspension mirror, telescope and scale. The scale is divided to millimetres and is about 3 metres from the mirror. The double-suspension mirror is fully described in the *Phil. Trans. A*, 1891, p. 572. It is of course not essential to the method, but was chosen because of the great magnification of the deflexion which it gives.

Let us suppose that the value of the scale-divisions of the deflexion is to be determined by a movement of the rider from right to left. The two micrometer-screws are withdrawn so that neither is in contact with the rider, that on the left so far that the rider will not touch it in its subsequent travel. The beam is lowered and allowed to swing. Then the right-hand screw is advanced till it bears against the end of the rider and pushes it some small distance. The contact is seen to have occurred by the interference with freedom of swing, as watched in the telescope. The micrometer is then read. Let its reading be m_1 . It is then withdrawn a little

so as to leave the rider free, and the centre of swing C_1 is determined in the usual way from three successive turning points. Then the micrometer is advanced again so as to push the rod a little further, and its reading m_2 is taken. It is then withdrawn and the new centre of swing C_2 is taken. If $m_1 - m_2 = d$, $C_1 - C_2$ divisions deflexion are due to an addition of Rd/b to the left pan.

The right-hand micrometer may then be withdrawn and the left-hand micrometer may be brought into action in a similar manner, and so on, the two screws being used alternately.

The balance-case was fixed on a shelf and was enclosed in a tin-foiled wood box with wool loosely packed between box and case. The case and box were provided with plate-glass windows to view the mirror and the micrometer divisions. The following abstract of some determinations of sensibility will serve to show what accuracy may be attained :—

I. Rider German silver wire, 7.35 mgm.

Half length of beam, 20.272 cm.

10 determinations alternately left and right.

Mean travel of rider 2.4850 mm.

Mean deflexion 21.26 divisions.

Mean value for 20 divisions ... 0.0848 mgm.

The separate determinations range between

20 divisions = 0.0877

and 20 „ = 0.0824.

II. The same rider.

10 determinations alternately left and right.

Mean travel of rider 5.2713 mm.

Mean deflexion 45.47 divisions.

Mean value for 40 divisions ... 0.1681 mgm.

The separate determinations range between

40 divisions = 0.1722

and 40 „ = 0.1632.

III. Rider German silver wire, 189.05 mgm.

7 determinations alternately left and right.

Mean travel of rider 0.1764 mm.

Mean deflexion 38.70 divisions.

Mean value for 40 divisions ... 0.1691 mgm.

The separate determinations range between

40 divisions = 0.1709

and 40 „ = 0.1654.

IV. The same rider.

7 determinations alternately left and right

Mean travel of rider 0.3004 mm.

Mean deflexion 64.84 divisions.

Mean value for 60 divisions ... 0.2578 mgm.

The separate determinations range between

60 divisions = 0.2612

and 60 „ = 0.2518.

LXVI. *The Balance as a Sensitive Barometer.**By G. W. TODD, M.Sc.**

It occurred to the author while testing with Professor Poynting the accuracy of a new method of determining the sensibility of a balance, described on p. 926, in which a thin rod or fibre is used instead of the usual rider, that a balance might with proper precautions be converted into a very delicate barometer.

A difference in volume on the opposite sides of a balance will give rise to motions of the pointer when the density of the surrounding air changes. Small oscillations of the pointer, due chiefly to convection currents, become negligible if this difference in volume is sufficiently large. Alterations in the density of the air are produced by changes in pressure, in temperature, and in the percentage of aqueous vapour in it. By eliminating the effects of any two of these the variations in the other may be obtained. Thus the balance might be used either as a barometer, thermometer, or hygrometer.

* Read June 25, 1909.

that the temperature in the balance was slowly rising and that the covered junctions were "lagging."

After this the temperature of the air inside the balance was measured by thermojunctions to avoid errors due to lagging. One set of junctions was kept at a fairly constant temperature in a Dewar tube containing water, the temperature being indicated on a Beckmann thermometer.

An idea of the sensitiveness of such a balance-barometer may be obtained from the table given below and the corresponding micro-barograph. The observations given extended over half an hour. The Beckmann thermometer in the Dewar vessel indicated the same temperature throughout.

TABLE.

1° C. difference between the junctions in the balance and those in the Dewar vessel corresponded to 28·8 divisions on the galvanometer-scale.

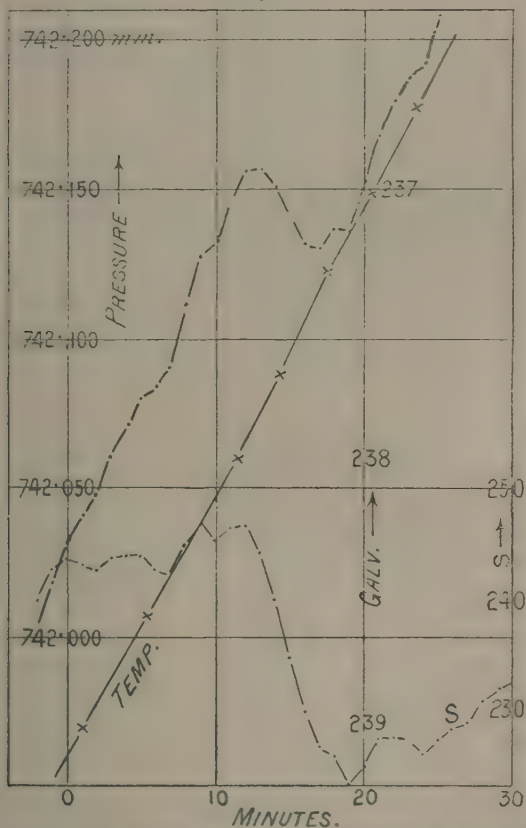
Galv. zero = 272·2. Room temp. = 13°·8 C. Rider at 13·152.

Time.	Telescope.	Galv.	Time.	Telescope.	Galv.
P.M. 4—0	243·7	239·1	P.M. 4—16	229·2	237·3
1	243·3		17	226·3	
2	242·6	238·9	18	225·6	
3	243·6		19	223·1	237·0
4	243·7	238·6	20	224·7	
5	243·9		21	227·0	
6	242·8		22	227·0	236·7
7	242·2	238·3	23	226·1	
8	244·9		24	225·8	
9	246·7	238·0	25	226·1	236·3
10	245·0		26	227·7	
11	246·2		27	228·5	236·0
12	246·3	237·7	28	230·9	
13	244·0		29	232·0	
14	240·1	234·6	30	232·6	232·1
15	234·6		31	232·1	

The second column in the table gives the telescope-readings of the balance-pointer, and the third gives the

galvanometer readings from which the temperature inside the balance is obtained. In the barograph the nearly straight curve represents the temperature, the lower irregular

Fig. 1.



curve the actual observations of the position of the balance-beam, and the upper curve the positions reduced to a constant temperature.

DISCUSSION.

Mr. M. E. J. GHERY expressed his interest in Mr. Todd's successful attempt to apply the well-known influence of barometric pressure upon the indications of a sensitive balance to detect small variations in the

atmospheric pressure. Prof. Marvin has just described (*Monthly Weather Review*, 1908) a delicate recording barometer, which shows plainly variations of $\cdot 05$ mm.; another type of recording barometer, the *statoscope*, records variations of the order of $\cdot 02$ mm. Both instruments give traces which show, even in calm weather, a succession of irregular ripples of variable amplitude and period. The question presents itself: can such an elaborate method as the one described by the Author be of practical value, when such minute changes can be actually recorded by instruments which require practically no attention besides the adjustment of the zero, and now and again a time check, and need no temperature correction?

Another point of greater importance is that when such small changes as these can be detected, there is no means of discriminating between the meteorological variations and the purely artificial ones, due to the fact that the instrument is in a building in which for various reasons there may be eddies and ascending or descending currents having nothing to do with the dynamics of the atmosphere, but leaving traces upon the record sheet. It is probable that a large number of the variations observed are due to entirely artificial changes of pressure, and such sensitive instruments, to give records of any use, should be placed in an isolated and quiet position, far from a building, in a shelter designed for the purpose.

It is in the study of the degree to which such sensitive barometers are affected by artificial perturbations that Mr. Todd's apparatus becomes useful. Such perturbations could be produced purposely and at regular intervals, and the magnitude of their effect on the instrument ascertained. To decide, therefore, if the records obtained so far by similar sensitive apparatus are the correct interpretation of genuine meteorological phenomena seems to be the only practical use to which Mr. Todd's ingenious apparatus can be put, but it is a very useful and necessary investigation to which much interest remains attached.

The AUTHOR in reply to Mr. Gheury recognized the practical value of a self-recording instrument. In the telescope of his balance-barometer one-tenth of a division could be estimated, so that the sensitiveness was from fifty to a hundred times greater than that of the *statoscope*. It therefore might be worth the little extra trouble involved in compounding two records—the temperature variation and the balance pointer variation—each of which could be made automatically.

The balance was little affected by sudden jars, such as the slamming of doors in other parts of the building. The opening or shutting of a door in the same room only occasionally caused a small motion, of the order of half a division, in the telescope. During observations this door was of course kept shut. A wave in the barograph of the same period as that of the balance might be due to a momentary impulse transmitted through the building, but other waves would not be open to the same suspicion.

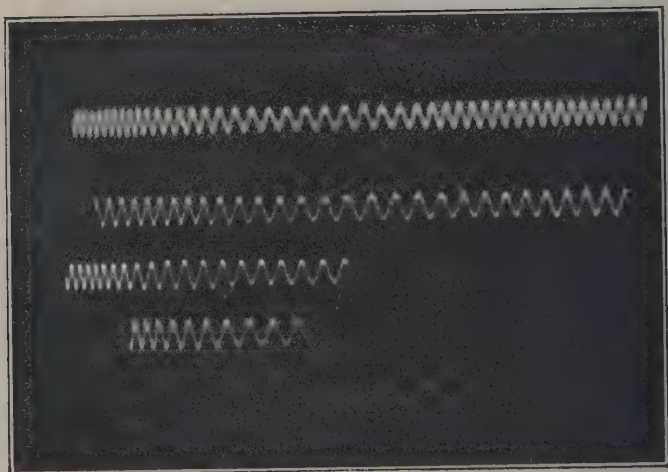


FIG. 7.



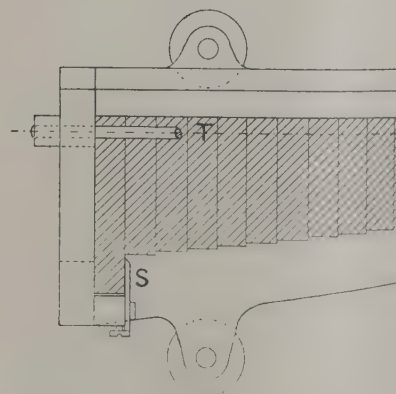
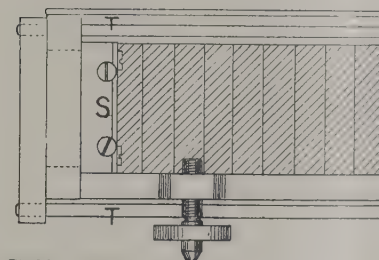
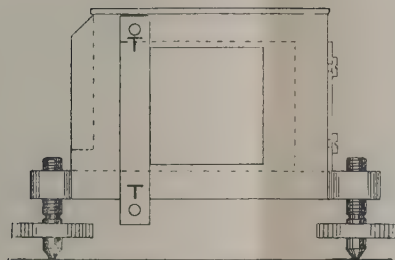
FIG. 8.

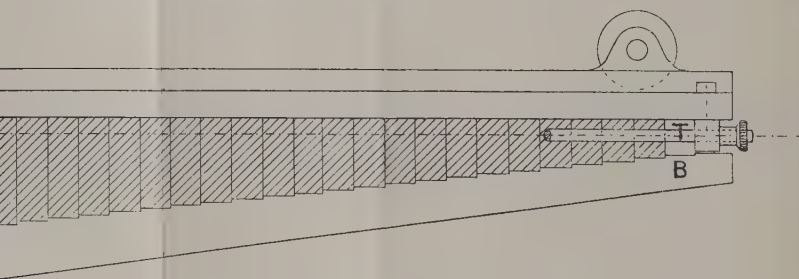
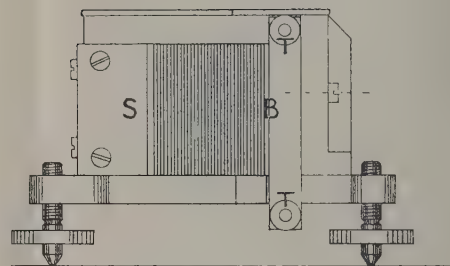
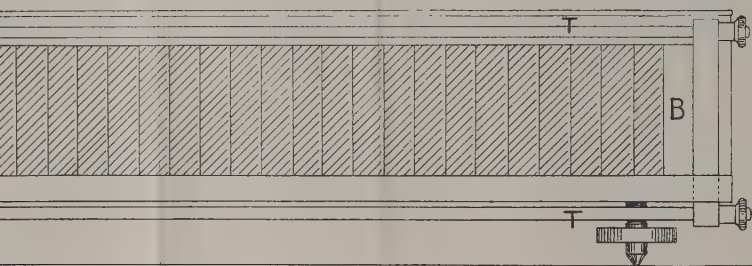


FIG. 7 a.

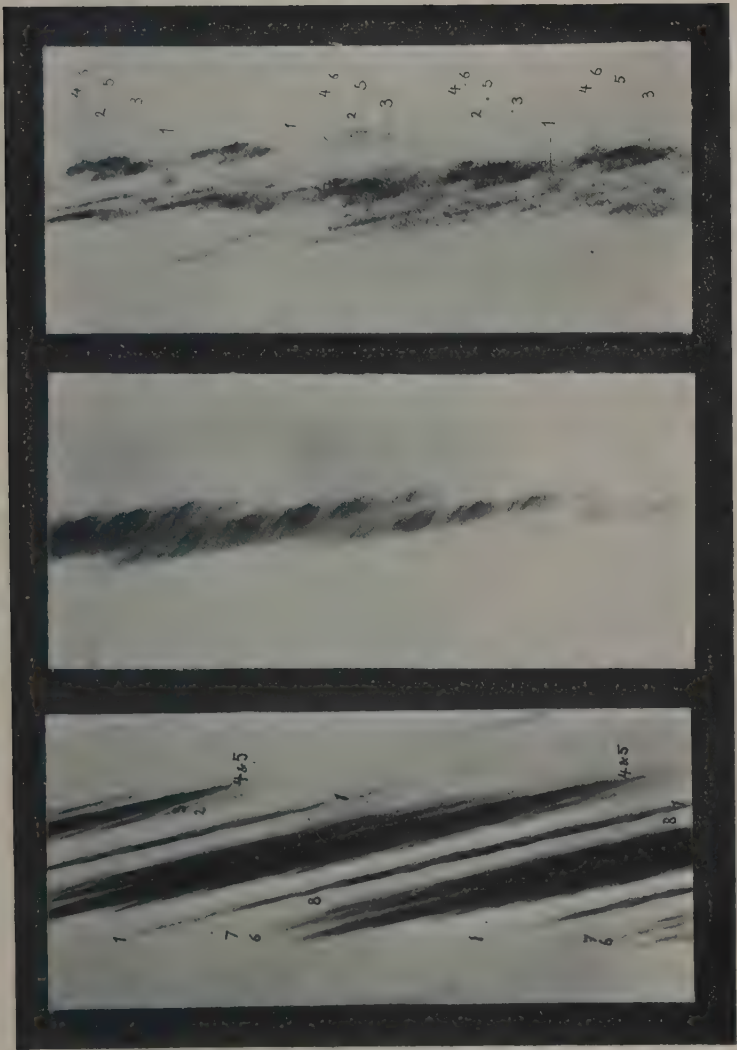


FIG. 9.





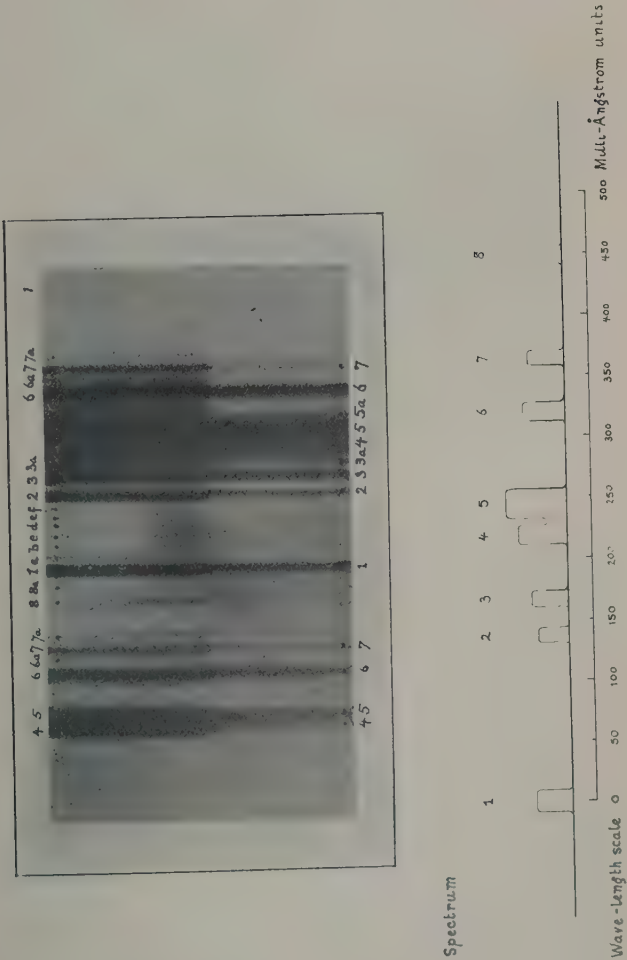
0 5 10 c.m



3.

2.

1.



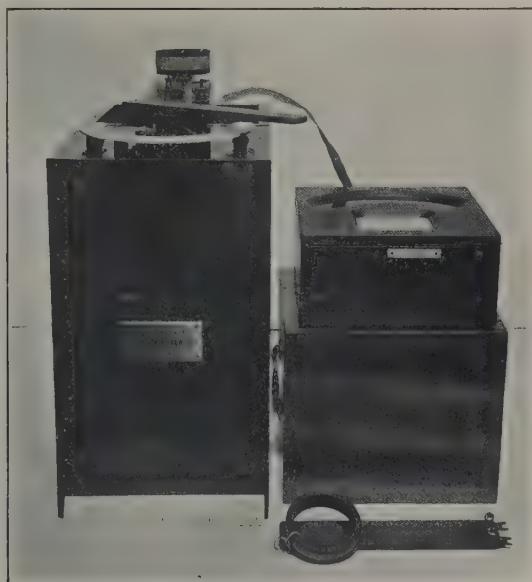


FIG. 1.



FIG. 2.

INDEX.

A.

	Page
Accurate method of measuring moments of inertia, on an	497
Acids, on the action between metals and	684
Action between metals and acids and the conditions under which mercury causes evolution of hydrogen, on the	684
Actinium and thorium emanations, on the diffusion of	485
Actinium, on the distribution in electric fields of the active deposits of radium, thorium, and	210
Air-blast upon the spark discharge of a condenser charged by an induction coil or transformer, on the effect of an	665
Allen, H. S., a graphic method of dealing with combinations of co- axial refracting surfaces	480
Alternate current circuits, on the use of the potentiometer on	561
Alternate current generator, on the theory of the	702
Alternate current measuring instruments, on the use of shunts and transformers with	235
Alternate currents in cables, on a vacuum-tube model for demon- strating the propagation of	297
Alternating currents suitable for telephonic and other measurements, on the production of small variable frequency	283
Alty, J. N., and Russell, A., an electro-magnetic method of studying the theory of and solving algebraical equations of any degree..	911
Amount of water in a cloud formed by expansion of moist air, note on the	444
Analogues of temperature equilibrium, on a note on certain dy- namical	224
Angle-bar section, on the least moment of inertia of an	653
Arc, observations on the electric	1
Archer, R. M., on a simple physical method of illustrating the principles of geometrical optics	442
Arthur Wright electrical device for evaluating formulæ and solving equations, on the	802

B.	Page
Balance as a sensitive barometer, on the	929
Balance, on a method of determining the sensibility of a	926
Barkla, C. G. and Sadler, C. A., homogeneous secondary Röntgen radiations	336
Barometer, on the balance as a sensitive	929
Bar magnets, on the self-demagnetizing factor of	622
Bars, on the lateral deflexion and vibration of "clamp-directed" ..	500
Bars supported at two points with one end overhanging, on the lateral vibration of	125
Battery, on a high-potential primary	640
Bellini, E., and Tosi, A., a directive system of wireless telegraphy .	305
Bending and twisting moments simultaneously, on a laboratory machine for applying	615
Bifilar vibration galvanometer, on a	774
Black body, note on the luminous efficiency of a	573
Bragg, W. H., and Glasson, J. L., on a want of symmetry shown by secondary X-rays	735
—, and Madsen, J. P. V., an experimental investigation of the nature of the γ -rays	261
Brush-discharge, on the effect of radiations on the	643
Bryan, G. H., a note on certain dynamical analogues of temperature equilibrium	224
Burton, C. V., a modified theory of gravitation	395
Butcher, D. D., experiments on artificial fulgurites	254

C.

Cadmium spectrum, method of producing an intense	892
Campbell, A., and Smith, T., on a method of testing photographic shutters	788
—, on the measurement of wave length for high frequency electrical oscillations	902
—, on the use of variable mutual inductances	69
Candle power, on the proposed international unit of	867
Cassie, W. R., an accurate method of measuring moments of inertia.	497
Charged particle in a combined electric and magnetic field, on an elementary treatment of the motion of a	300
Circuits, on the inductance and resistance in telephone and other ..	850
Clack, B. W., on the coefficient of diffusion	374
"Clamp-directed" bars, on the lateral deflexion and vibration of ..	500
Coefficient of diffusion, on the	374
Cohen, B. S., production of small variable frequency alternating currents suitable for telephonic and other measurements	283
Coker, E. G., a laboratory machine for applying, bending, and twisting moments simultaneously	615
Combinations of co-axial refracting surfaces, on a graphic method of dealing with,	480

	Page
Concentric main, on the effective resistance and inductance of a	581
Conditions under which mercury causes evolution of hydrogen, the .	684
Conductor of uniform thickness, on resistance of a	329
Contact potential differences determined by means of null solutions, on the	127
Cosine flicker photometer, on a form of	36
Current generator, on the theory of the alternate	702
Currents in cables, on a vacuum-tube model for demonstrating the propagation of alternate	297
Currents suitable for telephonic and other measurements, on the production of small variable frequency alternating	283
Curves, on observations on recalcence	180

D.

Deflexion and vibration of "clamped-directed" bars, on the lateral .	500
Device for evaluating formulæ and solving equations, on the Arthur Wright electrical	802
Diffusion of actinium and thorium emanations, on the	485
Diffusion, on the coefficient of	374
Directive system of wireless telegraphy, on a	305
Distribution in electric fields of the active deposits of radium, thorium, and actinium, on the	210
Dow, J. S., on a form of cosine flicker photometer	36
Drysdale, C. V., a vacuum-tube model for demonstrating the propa- gation of alternate currents in cables	297
—, note on the luminous efficiency of a black body	573
—, notes on the plug permeameter	229
—, the use of shunts and transformers with alternate current measuring instruments	235
—, the use of the potentiometer on alternate current circuits . . .	561
Duddell, W., on a bifilar vibration galvanometer	774
—, short spark phenomena	275
Dyke, G. B., and Fleming, J. A., note on the production of steady electric oscillations in closed circuits and a method of testing radiotelegraphic receivers	657
Dynamical analogues of temperature equilibrium, on a note on certain	224

E.

Eagle, A., on the form of the pulses constituting full radiation or white light	884
Echelon spectroscope, its secondary action, and the structure of the green mercury line	822
Effect of an air-blast upon the spark discharge of a condenser charged by an induction coil or transformer, on the	665
Effect of radiations on the brush-discharge, on the	643

	Page
Effect of temperature on the hysteresis loss in iron in a rotating field, on the	794
Effective resistance and inductance of a concentric main, on the ..	581
Electric arc, observations on the	1
Electric oscillations in closed circuits and a method of testing radiotelegraphic receivers, note on the production of steady ..	657
Electrical device for evaluating formulæ and solving equations, on the Arthur Wright	802
Electromagnetic method of studying the theory of and solving algebraical equations of any degree, on an	911
Elementary treatment of the motion of a charged particle in a combined electric and magnetic field, on an	300
Elliptic polarization produced by the direct transmission of a plane polarized stream through a plate of quartz, on the	548
Emanations, on the diffusion of actinium and thorium	485
Equilibrium, on a note on certain dynamical analogues of temperature	224
Equipotentials, on a freehand graphic way of determining stream lines and	88
Evolution of hydrogen, the conditions under which mercury causes.	684
Expansion of moist air, note on the amount of water in a cloud formed by	444
Experimental examination of Gibbs's theory of surface-concentration, on an	150
Experimental investigation of Gibbs's theory of surface-concentration, on an	515
Experiments on artificial fulgurites	254

F.

Fleming, J. A., note on the photoelectric properties of potassium-sodium alloy	469
——, on magnetic oscillators as radiators in wireless telegraphy ..	47
——, on some observations on the Poulsen arc as a means of obtaining continuous electrical oscillations	23
——, and Dyke, G. B., note on the production of steady electric oscillations in closed circuits and a method of testing radiotelegraphic receivers	657
——, and Richardson, H. W., effect of an air-blast upon the spark discharge of a condenser charged by an induction coil or transformer	665
Form of the pulses constituting full radiation or white light, on the.	884
Freehand graphic way of determining stream lines and equipotentials, on a	88
Fulgurites, on experiments on artificial	254
Fuller, W. P., and Grace, H., effect of temperature on the hysteresis loss in iron in a rotating field	794

G.

Page

Galvanometer, on a bifilar vibration	774
Garrett, A. E., the effect of radiations on the brush-discharge	643
Generator, on the theory of the alternate current	702
Geometrical optics, on a simple physical method of illustrating the principles of	442
Gibbs's theory of surface-concentration, on an experimental examination of	150
Gibbs's theory of surface-concentration, on an experimental investigation of	515
Glasson, J. L., and Bragg, W. H., on a want of symmetry shown by secondary X-rays	735
Grace, H., and Fuller, W. P., effect of temperature on the hysteresis loss in iron in a rotating field	794
Graphic method of dealing with combinations of co-axial refracting surfaces, on a	480
Gravitation, on a modified theory of	395

H.

High-potential primary battery, on a	640
Homogeneous secondary Röntgen radiations	336
Hysteresis loss in iron in a rotating field, on the effect of temperature on the	794

I.

Inductance and resistance in telephone and other circuits, on the ..	850
Inductance of two parallel wires, on the	447
Inductances, on the use of variable mutual	69
Inertia, on an accurate method of measuring moments of	497
Instrument for measuring the strength of an intense horizontal confined magnetic field, on an	922
Instruments, on the use of shunts and transformers with alternate current measuring	235
International unit of candle power, on the proposed	867
Investigation of the nature of the γ rays, on an experimental	261

J.

Jordan, F. W., an instrument for measuring the strength of an intense horizontal confined magnetic field	922
--	-----

L.

Lateral deflexion and vibration of "clamped-directed" bars, on the.	500
Lateral vibration of bars supported at two points with one end overhanging, on the	125

	Page
Least moment of inertia of an angle-bar section, on the	653
Lees, C. H., on the resistance of a conductor of uniform thickness whose breadth suddenly changes, and on the shapes of the stream-lines in the immediate neighbourhood	329
Lewis, W. C. M ^c C., on an experimental examination of Gibbs's theory of surface-concentration	150
—, an experimental investigation of Gibbs's theory of surface- concentration, regarded as the basis of adsorption	515
Lowry, T. M., method of producing an intense cadmium spectrum..	892
Luminous efficiency of a black body, note on the	573
Lyle, T. R., theory of the alternate current generator	702

M.

Machine for applying bending and twisting moments simultaneously, on a laboratory	615
Madsen, J. P. V., and Bragg, W. H., an experimental investigation of the nature of the γ rays	261
Magnetic field, on an instrument for measuring the strength of an intense horizontal confined	922
Magnetic oscillators as radiators in wireless telegraphy, on	47
Magnetism, note on terrestrial	890
Measurement of wave length for high frequency electrical oscilla- tions, on the	902
Measuring the self-inductance of a coil, on Pirani's method of	634
Metals and acids, on the action between	684
Method of dealing with combinations of co-axial refracting surfaces, on a graphic	480
Method of determining the sensibility of a balance, on a	926
Method of illustrating the principles of geometrical optics, on a simple physical	442
Method of measuring moments of inertia, on an accurate	497
Method of producing an intense cadmium spectrum	892
Method of studying the theory of and solving algebraical equations of any degree, on an electromagnetic	911
Method of testing photographic shutters, on a	788
Modified theory of gravitation, on a	395
Moment of inertia of an angle-bar section, on the least	653
Moments of inertia, on an accurate method of measuring	497
Morrow, J., on the lateral deflexion and vibration of "clamped- directed" bars	500
—, on the lateral vibration of bars supported at two points with one end overhanging	125
Morton, W. B., an elementary treatment of the motion of a charged particle in a combined electric and magnetic field	300
Morton, W. B., note on the amount of water in a cloud formed by expansion of moist air	444

	Page
Moss, E. W., and Thompson, S. P., on the self-demagnetizing factor of bar magnets	622
Moss, H., and Smith, S. W. J., on the contact potential differences determined by means of null solutions	127
Motion of a charged particle in a combined electric and magnetic field, on a elementary treatment of the	300

N.

Nature of the γ rays, on an experimental investigation of the	261
Nicholson, J. W., inductance and resistance in telephone and other circuits	850
——, the inductance of two parallel wires	447
Note on certain dynamical analogues of temperature equilibrium ..	224
Note on terrestrial magnetism	890
Note on the amount of water in a cloud formed by expansion of moist air	444
Note on the luminous efficiency of a black body, a	573
Note on the photoelectric properties of potassium sodium alloy	469
Null solutions, on the contact potential differences determined by means of	127

O.

Observations on recalescence curves, on	180
Observations on the electric arc	1
Observations on the Poulsen arc as a means of obtaining continuous electrical oscillations, on some	23
Optics, on a simple physical method of illustrating the principles of geometrical	442
Oscillations in closed circuits and a method of testing radiotelegraphic receivers, note on the production of steady electric	657
Oscillations, on the measurement of wave length for high frequency electrical	902

P.

Particle in a combined electric and magnetic field, on an elementary treatment of the motion of a charged	300
Patersen, C. C., the proposed international unit of candle power ..	867
Perneameter, notes on the plug	229
Phenomena, on short spark	275
Photographic shutters, on a method of testing	788
Photometer, on a form of cosine flicker	36
Pirani's method of measuring the self-inductance of a coil, on	634

	Page
Plug permeameter, notes on the	229
Polarization produced by the direct transmission of a plane polarized stream through a plate of quartz, on the elliptic	548
Potassium-sodium alloy, a note on the photoelectric properties of ..	469
Potential differences determined by means of null solutions, on the contact	127
Potentiometer on alternate current circuits, on the use of the	561
Poulsen arc as a means of obtaining continuous electrical oscillations, on some observations on the	23
Poynting, J. H., and Todd, G. W., on a method of determining the sensibility of a balance	926
Primary battery, on a high-potential	640
Production of small variable frequency alternating currents suitable for telephonic and other measurements	288
Production of steady electric oscillations in closed circuits and a method of testing radiotelegraphic receivers, note on the	657
Propagation of alternate currents in cables, on a vacuum-tube model for demonstrating the	297
Properties of potassium-sodium alloy, a note on the photoelectric ..	469
Pulses constituting radiation or white light, on the form of the	884

R.

Radiation or white light, on the form of the pulses constituting....	884
Radiations, on homogeneous secondary Röntgen	336
Radiations, on the brush-discharge, on the effect of	643
Radiotelegraphic receivers, method of testing	657
Radium, thorium, and actinium, on the distribution in electric fields of the active deposits of	210
Rays, on an experimental investigation of the nature of the γ	261
Recalescence curves, observations on	180
Refracting surfaces, on a graphic method of dealing with combinations of co-axial	480
Resistance and inductance of a concentric main, on the effective ..	581
Resistance of a conductor of uniform thickness whose breadth suddenly changes, and on the shapes of the stream-lines in the immediate neighbourhood	329
Richardson, H. W., and Fleming, J. A., effect of an air-blast upon the spark discharge of a condenser charged by an induction coil or transformer	665
Richardson, L. F., on a freehand graphic way of determining stream lines and equipotentials	88
Röntgen radiations, on homogeneous secondary	336
Rosenhain, W., on observations on recalescence curves	180
Rowell, H. S., on the least moment of inertia of an angle-bar section	653

Russ, S., on the distribution in electric fields of the active deposits of radium, thorium, and actinium	210
—, the diffusion of actinium and thorium emanations	485
Russell, A., the effective resistance and inductance of a concentric main, and methods of computing the ber and bei and allied functions	581
—, and Alty, J. N., an electromagnetic method of studying the theory of and solving algebraical equations of any degree . . .	911
—, and Wright, A., the Arthur Wright electrical device for evaluating formulæ and solving equations	802

S.

Sadler, C. A., transformation of X-rays	746
—, and Barkla, C. G., homogeneous secondary Röntgen radiations. .	336
Secondary Röntgen radiations, homogeneous, on	336
Self-demagnetizing factor of bar magnets, on the	622
Self-inductance of a coil, on Pirani's method of measuring the . . .	634
Sensibility of a balance, on a method of determining the	926
Sensitive barometer, on the balance as a	929
Sexton, F. P., the spectrum top	392
Short spark phenomena	275
Shunts and transformers with alternate current measuring instruments, on the use of	235
Simple physical method of illustrating the principles of geometrical optics, on a	442
Smith, S. W. J., on the action between metals and acids and the conditions under which mercury causes evolution of hydrogen .	684
—, and Moss, H., on the contact potential differences determined by means of null solutions	127
Smith, T., and Campbell, A., on a method of testing photographic shutters.	788
Snow, E. C., on Pirani's method of measuring the self-inductance of a coil.	634
Spark discharge of a condenser charged by an induction-coil or transformer, on the effect of an air-blast upon the	665
Spectroscope, the echelon	822
Spectrum, method of producing an intense cadmium	892
Spectrum top, the	392
Stansfield, H., the echelon spectroscope, its secondary action, and the structure of the green mercury line	822
Stream lines and equipotentials, on a freehand graphic way of determining	88
Surface-concentration, on an experimental examination of Gibb's theory of	150
Surface-concentration, on an experimental investigation of Gibb's theory of	515
Symmetry shown by secondary X-rays, on a want of	735

	Page
T.	
Telegraphy, on a directive system of wireless	305
Telegraphy, on magnetic oscillators as radiators in wireless.....	47
Telephone and other circuits, on the inductance and resistance in ..	850
Temperature equilibrium, on a note on certain dynamical analogues of	224
Temperature on the hysteresis loss in iron in a rotating field, on the effect of	794
Terrestrial magnetism, note on	890
Theory of gravitation, on a modified	395
Theory of the alternate current generator, on the	702
Thompson, S. P., and Moss, E. W., on the self-demagnetizing factor of bar magnets.....	622
Thorium and actinium, on the distribution in electric fields of the active deposits of radium	210
Thorium emanations, on the diffusion of actinium and	485
Todd, G. W., the balance as a sensitive barometer	929
—, and Poynting, J. H., on a method of determining the sensibility of a balance	926
Top, the spectrum	392
Tosi, A., and Bellini, E., a directive system of wireless telegraphy..	305
Transformation of X-rays, on	746
Tucker, W. S., a high-potential primary battery	640

U.

Unit of candle power, on the proposed international	867
Upson, W. L., on observations on the electric arc	1
Use of the potentiometer on alternate current circuits, on the	561
Use of variable mutual inductances, on the.....	69

V.

Vacuum-tube model for demonstrating the propagation of alternate currents in cables	297
Vibration galvanometer, on a bifilar.....	774
Vibration of bars supported at two points with one end overhanging, on the lateral	125
Vibration of "clamped-directed" bars, on the lateral deflexion and .	500

W.

Walker, G. W., note on terrestrial magnetism	890
Walker, J., on the elliptic polarization produced by the direct transmission of a plane polarized stream through a plate of quartz	548

INDEX.

945

Page

Want of symmetry shown by secondary X-rays, on a	735
Water in a cloud formed by expansion of moist air, note on the amount of	444
Wave length for high frequency electrical oscillations, on the measurement of	902
Wireless telegraphy, on a directive system of	305
Wright, A., and Russell, A., the Arthur Wright electrical device for evaluating formulæ and solving equations	802

X.

X-rays, on a want of symmetry shown by secondary	735
X-rays, on transformation of	746

